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Unavoidable minors of graphs of large type

John Dittmann[a], Bogdan Oporowski[b,*]

[a] Department of Mathematics, University of New Orleans, New Orleans, LA 70148, USA
[b] Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA

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Abstract

In this paper, we study one measure of complexity of a graph, namely its type. The *type* of a graph G is defined to be the minimum number n such that there is a sequence of graphs $G = G_0, G_1, \ldots, G_n$, where G_i is obtained by contracting one edge in or deleting one edge from each block of G_{i-1}, and where G_n is edgeless. We show that a 3-connected graph has large type if and only if it has a minor isomorphic to a large fan. Furthermore, we show that if a graph has large type, then it has a minor isomorphic to a large fan or to a large member of one of two specified families of graphs.

MSC: 05C83, 05C75

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1. Introduction

Graphs in this paper are finite and may have loops and multiple edges. A graph G is a *minor* of a graph H, written $G \leqslant_{\mathrm{m}} H$, if G can be obtained from a subgraph of H by contracting edges. The following is the celebrated Robertson–Seymour Theorem [4], previously known as Wagner's Conjecture.

Theorem 1.1. *Every infinite set of graphs contains two elements one of which is isomorphic to a minor of the other.*

One of the central problems in matroid theory lies in determining the classes of matroids that admit an extension of the Robertson–Seymour Theorem. More precisely, it addresses the following:

* Corresponding author.
E-mail addresses: dittmann@math.uno.edu (J. Dittmann), bogdan@math.lsu.edu (B. Oporowski).

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Question 1.2. *For which finite fields F does every infinite set of matroids representable over F contain two elements one of which is isomorphic to a minor of the other?*

While this problem seems very difficult, a small but significant step towards its solution has been made by one of the authors in collaboration with others [1] by proving that the Robertson–Seymour Theorem extends to every class of matroids that are representable over a fixed finite field and have an additional structural property: bounded type. The *type* $t(M)$ of a matroid M is defined as follows: If $E(M)=\emptyset$, then $t(M)=0$. If $E(M)\neq\emptyset$ and M is connected, then $t(M)=1+\min\{t(M\backslash e),t(M/e)\colon e\in E(M)\}$. If M is disconnected, then $t(M)=\max\{t(M_i)\}$, where the maximum is taken over all connected components M_i of M. The previously mentioned result from [1] may now be stated more precisely as follows:

Theorem 1.3. *Let F be a finite field, k be a nonnegative integer, and $\mathscr{M}$ be the class of F-representable matroids of type at most k. Then every infinite subset of $\mathscr{M}$ contains two elements one of which is isomorphic to a minor of the other.*

Since Theorem 1.3 answers Question 1.2 for classes of matroids of bounded type, it provides strong motivation to characterize classes of matroids of unbounded type. Informally speaking, we would like to describe such classes by the presence of certain minors in their members. We believe that finding such descriptions for the class of all matroids is very difficult, but that it becomes easier for more restricted classes of matroids. In this paper, we give such a characterization for graphic matroids.

Even though the results of this paper have been motivated by research in matroid theory, they speak of graphs. Consequently, for the reader's convenience, we translate the definition of type into the language of graph theory. If a graph G is edgeless, then its *type*, $t(G)$, is zero. If G has edges and G is a block, then $t(G)=1+\min\{t(G\backslash e),t(G/e)\colon e\in E(G)\}$. If G is not a block, then $t(G)=\max\{t(G_i)\}$, where the maximum is taken over all blocks G_i of G. Clearly, the definitions of type for matroids and for graphs are consistent, that is, the type of a graph equals the type of the cycle matroid of the graph.

In Section 2, we examine three families of graphs: fans, multicycles, and comulticycles. For a positive integer n, an *n-fan* F_n is the graph obtained from a path on n vertices by adding a new vertex and joining it to all vertices of the path. For integers m and n exceeding 2 and 0, respectively, the *(m,n)-multicycle* $C_{m,n}$ is the graph obtained from a cycle on m vertices by replacing each of its edges by n parallel edges. An *(m,n)-comulticycle* $C^*_{m,n}$ is the planar dual of $C_{m,n}$, or, equivalently, the graph obtained from an m-edge bond C^*_m (the graph consisting of m parallel edges joining two vertices) by subdividing each of the edges by $n-1$ new vertices. In Section 2 we show, roughly speaking, that if m and n are large, then so are the types of F_n, $C_{m,n}$, and $C^*_{m,n}$. We also show there that type is not monotone under the taking of minors, that is, that $G\leqslant_{\mathrm{m}} H$ does not imply $t(G)\leqslant t(H)$. We believe that this lack of monotonicity is the main reason for the difficulty in obtaining a desired characterization of classes of matroids of bounded type.

In Section 3, we characterize 3-connected graphs of large type. For a positive integer n, an n-wheel W_n is the graph obtained from a cycle on n vertices by adding a new vertex and joining it to all vertices of the cycle. Section 3 contains proof of the following two theorems.

Theorem 1.4. *If G is a graph that contains F_n as a minor, then the type $t(G)$ of G is at least $\lfloor \log_2 n \rfloor + 1$.*

Theorem 1.5. *For each integer n exceeding 2, there is an integer t_n such that if G is a 3-connected graph and $t(G) \geqslant t_n$, then G has a minor isomorphic to an n-spoke wheel W_n.*

Note that, since, for each n, the fan F_n is a minor of the wheel W_n, Theorems 1.4 and 1.5 imply the following, somewhat informal, remark, which describes the first main result of the paper.

Remark 1.6. A 3-connected graph has large type if and only if it has no minor isomorphic to a large fan.

The second of the main results of the paper is the following analog of Theorem 1.5 for arbitrary graphs.

Theorem 1.7. *For every integer n exceeding 3, there is a number N such that every graph whose type is at least N has a minor isomorphic to one of $\{F_n, C_{n,n}, C^*_{n,n}\}$.*

We note that, as will be shown in Section 2, graphs containing a minor isomorphic to $C_{n,n}$ or $C^*_{n,n}$ for a large value of n may have small type. Consequently, Theorem 1.4 has no analog for arbitrary graphs. From Section 4 onwards, the paper is devoted to proving Theorem 1.7.

The set of vertices of a graph G will be denoted by $V(G)$, and the set of edges of G by $E(G)$. An edge that is not a loop is a *link-edge*, and a nonempty maximal class of parallel link-edges is a *multi-edge*. If a multi-edge contains at least two edges, then the multi-edge is *proper*; otherwise, it is *trivial*. We shall write $e \,||\, uv$ to indicate that the endvertices of e are u and v. If v is a vertex of a graph G, then the *degree* of v in G is $|E_v| + 2|L_v|$, where E_v is the set of link-edges of G incident with v and L_v is the set of loops of G incident with v. If n is a nonnegative integer, then an *n-path* is a graph isomorphic to the path on $n+1$ vertices. A path of length 0 is *trivial*; otherwise it is *proper*.

If $e \in E(G)$, then we shall use the standard notation of $G\backslash e$ and G/e to denote the deletion of the edge e from G and the contraction of the edge e in G, respectively. Also, if $v \in V(G)$, then $G - v$ denotes the deletion of v (and the edges incident with v) from G.

If $E' \subseteq E(G)$ and $V' \subseteq V(G)$, then $G\backslash E'$, G/E', and $G - V'$ are defined in the obvious way. If H is a subgraph of G, then let $G/H = G/E(H)$, and let $G\backslash H = (G\backslash E(H)) - V_H$, where V_H is the set isolated vertices in $G\backslash E(H)$ whose elements are not isolated vertices in G.

If H can be obtained by deleting only vertices from G, then it is standard to say that H is an *induced subgraph* of G. If $V' \subseteq V(G)$, then $G[V']$ is the induced subgraph obtained by deleting $V(G) - V'$ from G. If $E' \subseteq E(G)$, then $G[E']$ is the smallest subgraph of G whose set of edges is E'. We shall use the notation $H \leqslant_{\mathrm{s}} G$ to denote that H is a subgraph of G. We say that H is a *topological minor* of G, denoted $H \leqslant_{\mathrm{t}} G$, if some subdivision of H is a subgraph of G. We write $G \cong H$ to indicate that the graphs G and H are isomorphic.

We say that a graph G is 2-*connected* if G is loopless, $|V(G)| + |E(G)| \geqslant 4$, and $G - v$ is connected, for each $v \in V(G)$. Equivalently, a graph G is 2-connected if and only if $|E(G)| \geqslant 2$ and each pair of edges of G is contained in a cycle of G. Also, we say that G is 3-*connected* if G is loopless, $|V(G)| \geqslant 4$, and $G - \{u, v\}$ is connected, for each pair $\{u, v\} \subseteq V(G)$. By a *block* of a graph G, we mean an isolated vertex of G, a loop of G, a cut-edge of G, or a maximal 2-connected subgraph of G.

Let H be a subgraph of a graph G. Then a *bridge* of H in G is one of the following kinds of subgraphs of G:

(i) An edge of $E(G) - E(H)$ contained in $G[V(H)]$.
(ii) The union of a component C of $G - V(H)$ and the set of edges that have one vertex in $V(C)$ and the other vertex in $V(H)$.

The disjoint union of sets or graphs will be denoted by $\dot{\cup}$. If k is a positive integer, then let $[k]$ denote the set of nonnegative integers less than $k + 1$, and let $[k]_+$ denote the set of positive integers less than $k + 1$.

2. Preliminary results

We begin by describing graphs of very small type. The following self-evident theorem characterizes graphs of type zero, one, and two.

Theorem 2.1. *Suppose G is a graph. The type of G is*

(i) *zero if and only if G is edgeless*;
(ii) *at most one if and only if G has no cycles other than loops*; *and*
(iii) *at most two if and only if every block of G is a multi-edge*, *a cycle*, *or an isolated vertex*.

As we remarked in Section 1, type is not monotone under the taking of minors. As a matter of fact, it is not monotone even under the taking of induced subgraphs. Consider the graph D in Fig. 1. The induced subgraph $D - v$ is isomorphic to $C^*_{5,3}$. We shall see later in Lemma 2.4 that $t(C^*_{5,3}) = 5$, but now we show that $t(D) \leqslant 4$. If the

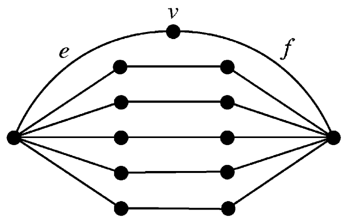

Fig. 1. D is the union of $C^*_{5,3}$, e, and f.

edges e and f are contracted in D, then the resulting graph consists of five 3-cycles meeting at a single vertex. Since each block of $D/\{e,f\}$ is a 3-cycle, $t(D/\{e,f\})=2$, by Theorem 2.1. Thus $t(D) \leqslant t(D/\{e,f\}) + |\{e,f\}| = 2+2=4$.

The graph in Fig. 1 can be easily modified to show that this lack of monotonicity under the taking of induced subgraphs is arbitrarily "bad" in the sense that there are graphs G and H such that G is an induced subgraph of H, but $t(G)-t(H)$ is arbitrarily large.

Although type does not have monotonicity under the taking of induced subgraphs, it does have some very special kinds of monotonicity that we shall describe below. Let G be a graph. The *simplification* of G, denoted $\tilde{G}$, is obtained by deleting the loops of G and by replacing each proper multi-edge of G with a link-edge. Now, let $\mathscr{C}$ be the collection of cycles in G each element of which has at most one vertex of degree exceeding two in G, and let $\mathscr{P}$ be the collection of proper paths P in G such that each internal vertex of P has degree 2 in G. Then the *cosimplification* of G is obtained by contracting all but one edge of each element of $\mathscr{C}$ and all but one edge of each maximal element of $\mathscr{P}$ in G.

A graph G is *simpler* than a graph H if G is a proper subgraph of H, and the simplifications of G and H are isomorphic. A graph G is *cosimpler* than a graph H if G can be obtained by contracting a non-empty set of edges in H, and the cosimplifications of G and H are isomorphic; equivalently, G is cosimpler than H if H can be obtained by subdividing each edge in a non-empty subset of $E(G)$ with at least one new vertex.

Lemma 2.2. *If G is simpler or cosimpler than H, then $t(G) \leqslant t(H)$.*

Proof. We shall only consider the case when G is simpler than H; the proof in the other case is very similar and left for the reader. Clearly, we may assume that $|E(H)| = |E(G)| + 1$. We proceed by induction on $t(H)$.

If $t(H)=1$, then the claim follows from Theorem 2.1. Suppose now that $t(G') \leqslant t(H')$ whenever G' is a graph that is simpler than H' with one edge fewer than H' and $t(H') < t(H)$. Let e denote the edge of H that is not in G. The proof in the case when e is a loop is trivial. Hence, we may assume that e is parallel to some edge e_G of G. Let B_H denote the block of H containing e and e_G, and let B_G denote the block of G containing e_G. Since each block in G different from B_G is a

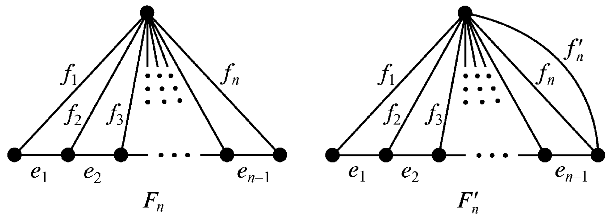

Fig. 2.

block in H, it is sufficient to show that $t(B_G) \leqslant t(B_H)$. Let e' be an edge of B_H such that $t(B_H) = \min\{t(B_H \backslash e'), t(B_H/e')\} + 1$. Then $t(B_H \backslash e') < n$ or $t(B_H/e') < n$. If e' is not parallel to e, then $B_G \backslash e'$ and B_G/e' are simpler than $B_H \backslash e'$ and B_H/e', respectively. Since $t(B_H \backslash e') < n$ or $t(B_H/e') < n$, it follows from the induction hypothesis that $t(B_G \backslash e') \leqslant t(B_H \backslash e')$ or $t(B_G/e') \leqslant t(B_H/e')$, and so $t(B_G) \leqslant t(B_H)$. Hence, we may assume that $e' = e$ or e' is parallel to e. If $t(B_H) = t(B_H \backslash e') + 1$, then $B_H \backslash e'$ is isomorphic to B_G, and consequently, $t(B_G) < t(B_H)$. If $t(B_H) = t(B_H/e') + 1$, then B_G/e_G is simpler than B_H/e', and we conclude from the induction hypothesis that $t(B_G/e_G) \leqslant t(B_H/e')$. Hence, $t(B_G) \leqslant t(B_H)$ and thus $t(G) \leqslant t(H)$. □

Now, we turn our attention to basic graphs of large type: fans, multicycles and comulticycles.

Lemma 2.3. *For every positive integer n, the type of the n-fan is* $\lceil \log_2 n \rceil + 1$.

Proof. Let us consider the *augmented n-fan* F'_n, which is the graph obtained by adding an edge f'_n that is parallel to f_n, where f_n is the edge of F_n as illustrated in Fig. 2.

Note that, by Lemma 2.2, we have $t(F_n) \leqslant t(F'_n) \leqslant t(F_{n+1})$, for each positive integer n, since F_n is simpler than F'_n, and F'_n is cosimpler than F_{n+1}. In particular, it follows that $t(F_m) \leqslant t(F_n)$ and $t(F'_m) \leqslant t(F'_n)$ whenever m and n are positive integers and $m \leqslant n$.

We shall proceed by induction on n. In addition to showing that $t(F_n) = \lceil \log_2 n \rceil + 1$, we shall also show that $t(F'_n) = \lceil \log_2 (n+1) \rceil + 1$.

If $n = 1$, the proof is clear. Now, assume that n is an integer exceeding 1 and that $t(F_{n'}) = \lceil \log_2 n' \rceil + 1$ and $t(F'_{n'}) = \lceil \log_2 (n'+1) \rceil + 1$, for each positive integer n' less than n.

First, we show that $t(F_n) \leqslant \lceil \log_2 n \rceil + 1$ and $t(F'_n) \leqslant \lceil \log_2 (n+1) \rceil + 1$. Consider $F_n \backslash e_{\lceil n/2 \rceil}$ and $F'_n \backslash e_{\lceil n/2 \rceil}$. The graph $F_n \backslash e_{\lceil n/2 \rceil}$ consists of a block that is isomorphic to $F_{\lceil n/2 \rceil}$ and another block that is isomorphic to $F_{\lfloor n/2 \rfloor}$. It follows that $t(F_n) \leqslant t(F_{\lceil n/2 \rceil}) + 1 = (\lceil \log_2 \lceil n/2 \rceil \rceil + 1) + 1 = \lceil \log_2 n \rceil + 1$. Similarly, $F'_n \backslash e_{\lceil n/2 \rceil}$ consists of a block that is isomorphic to $F_{\lceil n/2 \rceil}$ and another block that is isomorphic to $F'_{\lfloor n/2 \rfloor}$. If n is even, then $F_{\lceil n/2 \rceil} = F_{n/2}$ is simpler than $F'_{\lfloor n/2 \rfloor} = F'_{n/2}$. It follows that $t(F'_n) \leqslant t(F'_{n/2}) + 1 = (\lceil \log_2 (n/2 + 1) \rceil + 1) + 1 = (\lceil \log_2 (n+2) - 1 \rceil + 1) + 1 = \lceil \log_2 (n+1) \rceil + 1$, when n is even. On the other hand, if n is odd, then $F'_{\lfloor n/2 \rfloor} = F'_{(n-1)/2}$ is cosimpler than $F_{\lceil n/2 \rceil} = F_{(n+1)/2}$. It follows that if n is odd, then $t(F'_n) \leqslant t(F_{(n+1)/2}) + 1 = (\lceil \log_2((n+1)/2) \rceil + 1) + 1 = (\lceil \log_2(n+1) - 1 \rceil + 1) + 1 = \lceil \log_2(n+1) \rceil + 1$.

It remains to show that $t(F_n) \geqslant \lceil \log_2 n \rceil + 1$ and $t(F'_n) \geqslant \lceil \log_2(n+1) \rceil + 1$. Since the proofs of these inequalities are very similar, we only prove the first one, while leaving the other to the reader. If e_i, f_1, or f_n is deleted from F_n, where $i \in [n-1]_+$, then the resulting graph consists of two blocks, one of which is isomorphic to $F_{n'}$, for some integer n' satisfying $\lceil n/2 \rceil \leqslant n' < n$, and so $t(F_{n'}) \geqslant t(F_{\lceil n/2 \rceil}) = \lceil \log_2 \lceil n/2 \rceil \rceil + 1 = \lceil \log_2 n \rceil$. Thus $t(F_n \backslash e_i) \geqslant \lceil \log_2 n \rceil$ and $t(F_n \backslash f_j) \geqslant \lceil \log_2 n \rceil$ for each $i \in [n-1]_+$ and for each $j \in \{1, n\}$. If f_i is deleted from F_n, where i is an integer satisfying $1 < i < n$, then F_{n-1} is cosimpler than the resulting graph $F_n \backslash f_i$; hence, $t(F_n \backslash f_i) \geqslant t(F_{n-1}) = \lceil \log_2(n-1) \rceil + 1 \geqslant \lceil \log_2 n \rceil$, for each i satisfying $1 < i < n$. If e_i, f_1, or f_n is contracted in F_n, where $i \in [n-1]_+$, then F_{n-1} is simpler than the resulting graph F_n/e, where $e \in \{e_i : i \in [n-1]_+\} \cup \{f_1, f_n\}$, and hence $t(F_n/e) \geqslant t(F_{n-1}) \geqslant \lceil \log_2 n \rceil$, for each $e \in \{e_i : i \in [n-1]_+\} \cup \{f_1, f_n\}$. If f_i is contracted in F_n, where i is an integer satisfying $1 < i < n$, then the resulting graph F_n/f_i consists of two blocks, one of which is isomorphic to $F'_{n'}$ for some integer n' satisfying $\lfloor n/2 \rfloor \leqslant n' < n$. Hence, $t(F_n/f_i) \geqslant t(F'_{\lfloor n/2 \rfloor}) = \lceil \log_2(\lfloor n/2 \rfloor + 1) \rceil + 1 \geqslant \lceil \log_2 n/2 \rceil + 1 = \lceil \log_2 n \rceil$, for each integer i satisfying $1 < i < n$. Thus, $t(F_n \backslash e) \geqslant \lceil \log_2 n \rceil$ and $t(F_n/e) \geqslant \lceil \log_2 n \rceil$, for each $e \in E(F_n)$. Consequently, the induction hypothesis implies that $t(F_n) \geqslant \lceil \log_2 n \rceil + 1$ for any positive integer n. $\square$

Lemma 2.4. *For every integer n exceeding* 3, *each of the* $(n, n-2)$*-multicycle and the* $(n, n-2)$*-comulticycle has type n.*

Proof. We prove a more general statement regarding the type of $C_{n,\geqslant m}$, where m and n are integers exceeding 1 and 3, respectively, and $C_{n,\geqslant m}$ represents any graph obtained by replacing one edge of C_n with a multi-edge containing exactly m edges and by replacing each of the remaining edges of C_n with a multi-edge containing at least m edges. The result that we prove here is that $t(C_{n,\geqslant m}) = \min\{n, m+2\}$. It will follow immediately that $t(C_{n,n-2}) = n$, for each integer n exceeding 3.

Let E_1 be a multi-edge of $C_{n,\geqslant m}$ consisting of m edges. Then each block of $C_{n,\geqslant m} \backslash E_1$ is a multi-edge. It follows from Theorem 2.1 that $t(C_{n,\geqslant m} \backslash E_1) \leqslant 2$; hence, $t(C_{n,\geqslant m}) \leqslant |E_1| + t(C_{n,\geqslant m} \backslash E_1) \leqslant m+2$. Let E_2 be a set of $n-2$ edges of $C_{n,\geqslant m}$ such that if f and f' are distinct edges in E_2, then f and f' belong to distinct multi-edges of $C_{n,\geqslant m}$. Then $C_{n,\geqslant m}/E_2$ consists of a block that is a multi-edge and blocks that are loops. It follows from Theorem 2.1 that $t(C_{n,\geqslant m}/E_2) \leqslant 2$; hence, $t(C_{n,\geqslant m}) \leqslant |E_2| + t(C_{n,\geqslant m}/E_2) \leqslant n$. Thus, $t(C_{n,\geqslant m}) \leqslant \min\{n, m+2\}$.

To prove the opposite inequality, we proceed by induction on m and n. It is easy to check that $t(C_{n,\geqslant 2}) = t(C_{4,\geqslant m}) = 4$ for integers m and n exceeding, respectively, 1 and 3. Now, let us assume that if m' and n' are integers satisfying $2 \leqslant m' < m$ and $4 \leqslant n' < n$, then $t(C_{n,\geqslant m'}) = \min\{n, m'+2\}$, and $t(C_{n',\geqslant m}) = \min\{n', m+2\}$. Consider the graph $C_{n,\geqslant m}$. For any edge $e \in E(C_{n,\geqslant m})$, the graph $C_{n,\geqslant m}/e$ is the union of a block that is a graph $C_{n-1,\geqslant m}$ and blocks that are loops. It follows from the induction hypothesis that $t(C_{n-1,\geqslant m}) = \min\{n-1, m+2\}$. If e belongs to a multi-edge of $C_{n,\geqslant m}$ that contains exactly m edges, then $C_{n,\geqslant m} \backslash e$ is a graph $C_{n,\geqslant m-1}$. It follows from the induction hypothesis that $t(C_{n,\geqslant m-1}) = \min\{n, m+1\}$. If e belongs to a multi-edge of $C_{n,\geqslant m}$ that contains more than m edges and f belongs to a multi-edge of $C_{n,\geqslant m}$ that contains exactly m edges, then $C_{n,\geqslant m} \backslash \{e, f\}$ is a graph $C_{n,\geqslant m-1}$ that is simpler than $C_{n,\geqslant m} \backslash e$. By Theorem 2.1 and the induction hypothesis, $t(C_{n,\geqslant m} \backslash e) \geqslant t(C_{n,\geqslant m-1}) = \min\{n, m+1\}$ when e belongs to a multi-edge containing more than m edges. It follows that $t(C_{n,\geqslant m}) = \min\{\min\{n-1, m+2\}, \min\{n, m+1\}\} + 1 = \min\{n-1, m+1\} + 1 = \min\{n, m+2\}$, as required.

The proof for comulticycles can be easily obtained by using the concept of duality. We leave the details to the reader. □

3. 3-Connected graphs

Although type is not monotone under the taking of minors, we can describe 3-connected graphs of large type as those without large fan minors. This characterization appears as Theorems 1.4 and 1.5. The entire remainder of this section will be devoted to proving these results. We start with some lemmas.

Lemma 3.1. *If n is an integer exceeding* 1, *and $F_n \leqslant_{\mathrm{m}} G$, then G contains a vertex set $S = \{v_i\colon i \in [n]_+\}$, a $v_1 v_n$-path P, and a tree T whose set of leaves is S, such that* $P \cap T = S$ and $F_n \leqslant_{\mathrm{m}} P \cup T$.

Proof. If $F_n \leqslant_{\mathrm{m}} G$, then there are disjoint subsets E_d and E_c in $E(G)$ such that $F_n \cong (G \backslash E_d / E_c)_E$, where H_E denotes the subgraph of a graph H obtained by deleting all isolated vertices from H; equivalently, $H_E = H[E(H)]$. Among all pairs (E_d, E_c) of disjoint sets of edges of $E(G)$ such that $F_n \cong (G \backslash E_d / E_c)_E$, choose one for which $|E_c|$ is minimum, and denote this pair by (D, C). Let $G' = (G \backslash D)_E$. A typical G' is illustrated in Fig. 3, where the dashed edges and the solid edges form, respectively, a path P and a tree T, and F_8 is obtained by contracting the unshaded edges.

If C is empty, then $G' \cong F_n$, and S, P, and T are obvious. So, we may assume that $C = \{e_i\colon i \in [k]_+\}$, for some positive integer k. Let the sequence $(e_i)_{i=1}^{k}$ be an arbitrary

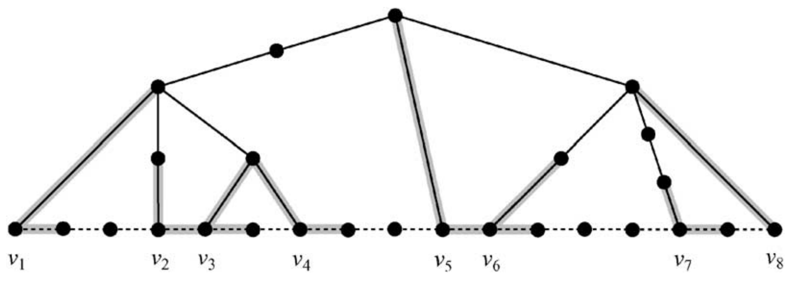

Fig. 3. A typical G' that contains an F_8-minor.

ordering of the elements of C. Let $G_0 = G'$, and, inductively, let $G_i = (G_{i-1}/e_i)_E$ for each $i \in [k]_+$, so that $G_k \cong F_n$. Note that since G_k is a block, $E(G_k)$ is contained in a single block B_i of G_i, for each $i \in [k]$. Moreover, G_i is a block for each $i \in [k]$, that is, $G_i = B_i$. This can be seen as follows. If G_i contained a block $B_i' \neq B_i$, then $(D \cup E(B_i'), C - E(B_i'))$ would be a pair of disjoint sets of edges such that $(G\backslash(D \cup E(B_i'))/(C - E(B_i')))_E = F_n$, but, since G_i has no isolated vertices (hence, $E(B_i')$ is nonempty), $|C - E(B_i')| < |C|$; a contradiction to the minimality of C.

Now, we show that $G' = P \cup T$. We proceed by induction on j to prove that, for each $j \in [k]$, the graph G_{k-j} contains a v_1v_n-path P_{k-j} and a tree T_{k-j} whose set of leaves is S, such that $P_{k-j} \cap T_{k-j} = S$. If $j = 0$, then $k - j = k$, and, since $G_k \cong F_n$, it is obvious what S, P_k, and T_k are. Assume that G_{k-j} contains subgraphs P_{k-j} and T_{k-j} that have the required properties, for each nonnegative integer $j < k$, and let $i = k - j$. Let v denote the vertex of G_i obtained by contracting e_i in G_{i-1}.

First, if $v \in V(P_i) - S$, then v is incident in G_i with exactly two edges e and f, which lie in P_i. After expanding v in G_i to e_i to obtain G_{i-1}, since G_{i-1} is a block, e_i is neither a loop nor a cut-edge in G_{i-1}. It follows that $G_{i-1}[E(P_i) \cup e_i]$ is a v_1v_n-path that contains the subpath ee_if. Let $P_{i-1} = G_{i-1}[E(P_i) \cup e_i]$ and $T_{i-1} = T_i$. It is straightforward that P_{i-1} and T_{i-1} have the required properties.

Now, if $v \in V(T_i) - S$, then v is incident in G_i with only edges in T_i. After expanding v in G_i to e_i to obtain G_{i-1}, since G_{i-1} is a block, e_i is neither a loop nor a cut-edge. Hence $G_{i-1}[E(T_i) \cup e_i]$ is a tree whose set of leaves is S. Let $P_{i-1} = P_i$ and $T_{i-1} = G_{i-1}[E(T_i) \cup e_i]$. It follows that P_{i-1} and T_{i-1} have the required properties.

Finally, assume that $v \in S$. Then either v is not an endvertex of P, or $v \in \{v_1, v_n\}$. We consider the case in which v is not an endvertex of P; the proof when $v \in \{v_1, v_n\}$, which is very similar, is left for the reader. If v is not an endvertex of P, then v is incident in G_i with exactly three edges e, f, and g, where $\{e, f\} \subseteq E(P_i)$ and $g \in E(T_i)$. After expanding v in G_i to e_i to obtain G_{i-1}, since G_{i-1} is a block, e_i is neither a loop nor a cut-edge. It follows that one vertex of e_i is trivalent in G_{i-1}; call this vertex v. Then one of the following holds for G_{i-1}.

(i) $G_{i-1}[E(P_i) \cup e_i]$ is a v_1v_n-path that contains the subpath ee_if, in which case, $P_{i-1} = G_{i-1}[E(P_i) \cup e_i]$ and $T_{i-1} = G_{i-1}[E(T_i)]$ have the required properties.

(ii) $G_{i-1}[E(T_i) \cup e_i]$ is a tree whose set of leaves is S, in which case, $P_{i-1} = P_i$ and $T_{i-1} = G_{i-1}[E(T_i) \cup e_i]$ have the required properties.

If $v \in \{v_1, v_n\}$, then v is incident in G_i with exactly two edges $e \in E(P_i)$ and $f \in E(T_i)$. If v is expanded in G_i to e_i to obtain G_{i-1}, then ee_if is a subpath in G_{i-1}. Let v denote the vertex in G_{i-1} common to e_i and f. It follows that the graphs $P_{i-1} = G_{i-1}[E(P_i) \cup e_i]$ and $T_{i-1} = T_i$ have the required properties. □

Lemma 3.2. *If n is an integer exceeding* 1, *and $F_n \leqslant_{\mathrm{m}} G$, then $F_{\lceil n/2 \rceil} \leqslant_{\mathrm{m}} G\backslash e$ for every $e \in E(G)$.*

Proof. Assume that $F_n \leqslant_{\mathrm{m}} G$. Then there are subgraphs P and T of G that satisfy the requirements specified in Lemma 3.1. If $e \notin P \cup T$, then $P \cup T$ is a subgraph of $G\backslash e$, and hence $F_{\lceil n/2 \rceil} \leqslant_{\mathrm{s}} F_n \leqslant_{\mathrm{m}} P \cup T \leqslant_{\mathrm{s}} G\backslash e$. If $e \in P$, then one component P' of $P\backslash e$ is a subpath of P containing at least $\lceil n/2 \rceil$ vertices in S, and hence $F_{\lceil n/2 \rceil} \leqslant_{\mathrm{m}} P' \cup T \leqslant_{\mathrm{s}} G\backslash e$. If $e \in T$, then one component T' of $T\backslash e$ is a subtree of T containing at least $\lceil n/2 \rceil$ vertices in S, and hence, $F_{\lceil n/2 \rceil} \leqslant_{\mathrm{m}} P \cup T' \leqslant_{\mathrm{s}} G\backslash e$. □.

Lemma 3.3. *If n is an integer exceeding* 1, *and $F_n \leqslant_{\mathrm{m}} G$, then $F_{\lfloor n/2 \rfloor} \leqslant_{\mathrm{m}} G/e$ for every $e \in E(G)$.*

Proof. Assume that $F_n \leqslant_{\mathrm{m}} G$. Then there are subgraphs P and T of G that satisfy the requirements stated in Lemma 3.1. If e is a loop, or if some vertex of e does not lie in $P \cup T$, then $F_{\lfloor n/2 \rfloor} \leqslant_{\mathrm{s}} F_n \leqslant_{\mathrm{m}} P \cup T \leqslant_{\mathrm{s}} G/e$. So we may assume that $e \,||\, xy$, and x and y are distinct vertices of $P \cup T$.

We shall use the following notation in proving this lemma. If $\{u, v\} \subseteq V(P)$, then let P_{uv} denote the uv-subpath of P, and, for each $v \in S$, let e_v denote the edge of T incident with v. We shall consider several cases depending on the location of x and y.

Suppose first that $\{x, y\} \subseteq S$. Let P' be the element of $\{P_{xy} \cup e, (P\backslash P_{xy}) \cup e\}$ that contains at least as many vertices of S as the other, and let $S' = S \cap V(P')$. Let $m = |S'|$. Clearly, $m \geqslant \lceil n/2 \rceil + 1$. Contract e to x. It follows that P'/e contains a path P'' having $m - 1$ vertices of $S' - y$. Since $(G/e)[E(T)]$ is obtained from T by identifying x and y, it follows that $\bigcup_{v \in S' - y} e_v$ is acyclic in G/e, and hence, there is a tree T' in $(G/e)[E(T)]$ that contains $\bigcup_{v \in S' - y} e_v$. Furthermore, $P'' \cap T' = S' - y$, and the set of leaves of T' contains $S' - y$. It follows that $F_{m-1} \leqslant_{\mathrm{m}} G/e$, and, since $m - 1 \geqslant \lceil n/2 \rceil$, that $F_{\lfloor n/2 \rfloor} \leqslant_{\mathrm{s}} F_{m-1} \leqslant_{\mathrm{m}} P'' \cup T' \leqslant_{\mathrm{s}} G/e$.

The case when $\{x, y\} \subseteq P$, but $\{x, y\} \cap S \leqslant 1$ is very similar to the one presented above, so we leave the details to the reader.

Suppose now that $e \,||\, xy$ has both vertices in $T - S$. Then P is a path in G/e. It is clear that $\bigcup_{v \in S} e_v$ is acyclic in G/e, and, since $(G/e)[E(T) - e]$ is connected, $(G/e)[E(T) - e]$ contains a tree T' whose set of leaves is $S = P \cap T'$. It follows that $F_{\lfloor n/2 \rfloor} \leqslant_{\mathrm{s}} F_n \leqslant_{\mathrm{m}} P \cup T' \leqslant_{\mathrm{s}} G/e$.

It remains to consider the case when one of x and y, say x, is in P and the other, y, is in $T - S$. Then there are not more than four edges of $P \cup T \cup e$ incident with x. One of these edges is e, and there are two distinct edges e^x and f^x of P incident with x. Consider the graph $(P \cup T \cup e)/e\backslash\{e^x, f^x\}$. One component P' of $(P \cup e)/e\backslash\{e^x, f^x\}$ is a path that contains at least $\lfloor n/2 \rfloor$ vertices of S. Let S' denote $V(P') \cap S$. It follows that $\bigcup_{v \in S'} e_v$ is acyclic in G/e. Since $(T \cup e)/e$ is connected, $(T \cup e)/e$ contains a tree T' whose set of leaves is $S' = P' \cap T'$. Then $F_{\lfloor n/2 \rfloor} \leqslant_{\mathrm{m}} P' \cup T' \leqslant_{\mathrm{s}} (P \cup T \cup e)/e\backslash\{e^x, f^x\} \leqslant_{\mathrm{s}} G/e$. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Suppose the theorem fails and let $\mathscr{G}$ be the collection of counterexamples to Theorem 1.4, that is, $\mathscr{G}$ consists of graphs H for which there is a positive integer $n(H)$ such that $F_{n(H)} \leqslant_{\mathrm{m}} H$, but $t(H) < \lfloor \log_2 n(H) \rfloor + 1$. Let $\mathscr{G}_0$ be the subcollection of $\mathscr{G}$ each of whose elements contain a minimum number of edges and no isolated vertices, and let $n = \min\{n(H): H \in \mathscr{G}_0\}$. Then $\mathscr{G}_0$ contains a graph G such that $n(G) = n$. Any such G is a minimal counterexample to Theorem 1.4 in the sense defined above. Note that $n \geqslant 2$ since if F_1 is a minor of a graph H, then $t(H) \geqslant \lfloor \log_2 1 \rfloor + 1 = 1$. It follows that since $F_n \leqslant_{\mathrm{m}} G$, there are subgraphs P and T of G that have the properties specified in Lemma 3.1.

The minimality of G implies that G is a block, and hence $\min\{t(G\backslash e), t(G/e)\} = t(G) - 1$. First, suppose that there is an edge e such that $t(G\backslash e) = t(G) - 1$. By Lemma 3.2, $F_{\lceil n/2 \rceil} \leqslant_{\mathrm{m}} G\backslash e$. Since $G\backslash e$ is not a counterexample,

$$t(G\backslash e) \geqslant \left\lfloor \log_2 \left\lceil \frac{n}{2} \right\rceil \right\rfloor + 1 \geqslant \left\lfloor \log_2 \frac{n}{2} \right\rfloor + 1 = \lfloor \log_2 n - 1 \; \lfloor \log_2 n \rfloor$$

and hence $t(G) \geqslant \lfloor \log_2 n \rfloor + 1$; a contradiction. Thus, if G is a counterexample, then there is an edge e such that $t(G/e) = t(G) - 1$.

Let e be an edge such that $t(G/e) = t(G) - 1$. By Lemma 3.3, $F_{\lfloor n/2 \rfloor} \leqslant_{\mathrm{m}} G/e$, that is, $F_{n/2} \leqslant_{\mathrm{m}} G/e$ if n is even, and $F_{(n-1)/2} \leqslant_{\mathrm{m}} G/e$ if n is odd. Since G/e is not a counterexample,

$$t(G/e) \geqslant \begin{cases} \lfloor \log_2 \frac{n}{2} \rfloor + 1 = \lfloor \log_2 n \rfloor & \text{if } n \text{ is even; and} \\ \lfloor \log_2 \frac{n-1}{2} \rfloor + 1 = \lfloor \log_2 (n-1) \rfloor = \lfloor \log_2 n \rfloor & \text{if } n \text{ is odd.} \end{cases}$$

Hence, $t(G) \geqslant \lfloor \log_2 n \rfloor + 1$; a contradiction, which proves Theorem 1.4. □

We now focus on proving Theorem 1.5. One of the major tools in proving this theorem is the following result of Seymour [6] (see also [1]).

Theorem 3.4. *Let C be a largest circuit of a connected matroid M. Then the size of every circuit of M/C is less than $|C|$.*

We use this theorem to derive the following two corollaries, the first of which is evident.

Corollary 3.5. *Every two longest cycles in a 2-connected graph intersect in at least two vertices.*

Corollary 3.6. *If the type of a 2-connected graph G exceeds $N(N+1)/2$, for some integer N greater than 1, then G contains a cycle of length more than N.*

Proof. Let G be a 2-connected graph, and assume that a longest cycle C of G has length N. We shall show that the type $t(G)$ of G is at most $N(N+1)/2$ by induction on N.

If $N = 2$, then G is a multi-edge and, by Theorem 2.1, the claim is immediate. Now, assume that $N > 2$ and that if the length of a longest cycle in a 2-connected graph G' is N', for some $N' < N$, then $t(G') \leqslant N'(N'+1)/2$. After contracting C in G, every cycle of G/C has length less than N, by Theorem 3.4. In particular, each cycle of each block of G/C has length less than N. By hypothesis, the type of each block that is neither a loop nor a cut-edge of G/C does not exceed $(N-1)N/2$, and it is evident that the type of a block that is a loop or a cut-edge does not exceed $(N-1)N/2$. It follows that $t(G) \leqslant N+(N-1)N/2 = N(N+1)/2$, as required, since G/C was obtained by contracting N elements in G, and since $t(G/C) \leqslant (N-1)N/2$. □

The second important ingredient of the proof of Theorem 1.5 is the following result.

Theorem 3.7. *For each positive integer n exceeding two, there is an integer N such that if G is a 3-connected graph with a cycle on N vertices, then the n-wheel, W_n, is a minor of G.*

Although, to our knowledge, this theorem has not been explicitly stated in literature, there are a few papers from which its proof can be derived. Since showing such a derivation formally here would require a large amount of new terminology and notation, we instead refer the reader to two proofs in [3,2]. The first of these, the proof of (1.4) in [3], speaks of graphs, although the derivation of Theorem 3.7 from it is fairly technical. On the other hand, the derivation of Theorem 3.7 from the proof of Theorem 1.5 in [2] requires translating from the language of binary matroids to the language of graphs, but the technical details of the derivation are easier.

It is worth noting that the value of N as a function of n that can be obtained through either derivation is extremely large and believed to be very far from the best possible bound. We are now ready to present the proof of Theorem 1.5.

Proof of Theorem 1.5. For each integer exceeding 2, let $t_n = N(N-1)/2+1$, where N is the number depending on n from Theorem 3.7. If G is a 3-connected graph whose type is at least t_n, then, by Theorem 3.4, G contains a cycle of length at least N. The conclusion now follows immediately from Theorem 3.7 □

4. 2-Sums and tree structures

The remainder of the paper will be devoted to proving Theorem 1.7. The main idea of the proof is to decompose the graph into pieces that are either 3-connected, or have very simple structure. We shall use a decomposition that relies on a result of Tutte, which states that every 2-connected graph has a canonical decomposition into simple 3-connected graphs, cycles, and multi-edges. In this section, we shall describe this decomposition and prove its basic properties, while in the remainder of the paper, we shall use it to prove Theorem 1.7.

If G is a graph, E_0 is a subset of $E(G)$, and S is a set, then define a function $L_G : E_0 \to S \times (V(G) \times V(G)) : e \mapsto (s(e), (u(e), v(e)))$ so that for each e in E_0, $u(e)$

and $v(e)$ are the endvertices of e, and if $s(e)=s(f)$, then $e=f$. Intuitively, we may think of L_G as a function which assigns to each edge e in E_0 a label $s(e)$ and a direction where $u(e)$ and $v(e)$ are, respectively, the tail and the head of e. Frequently, we shall describe these functions in this intuitive way. Also, it is convenient to think of the function L_G on E_0 as a partial function $L_G : E(G) \to S \times (V(G) \times V(G))$, where $L_G(e)$ is defined if and only if $e \in E_0$; often, we shall consider such functions L_G without specifying the domain of definition. Call L_G a *directed labeling* of G. It is clear that restricting the domain of L_G to a subset $E' \subseteq E_0$ results in a directed labeling L'_G of G, which will be called a *restriction* of L_G. If the domain of L_G is the empty set, then call the directed labeling L_G of G *trivial*, and we may also say that G is *unlabeled*. It is also clear that if G' is a minor of G, then $L_{G'} : E(G') \cap E_0 \to S \times (V(G') \times V(G'))$: $e \mapsto (s(e), (u'(e), v'(e)))$ is a directed labeling of G', where $u'(e)$ and $v'(e)$ are the vertices in G' that correspond to $u(e)$ and $v(e)$, respectively, in $V(G)$. In such case call $L_{G'}$ the directed labeling of G' *induced by* L_G.

Assume that $L_H : E(H) \to S \times (V(H) \times V(H)) : e \mapsto (s(e), (u_H(e), v_H(e)))$ and $L_K : E(K) \to S \times (V(K) \times V(K)) : e \mapsto (s(e), (u_K(e), v_K(e)))$ are directed labelings of disjoint graphs H and K, respectively, and there is only one pair, $h \in E(H)$ and $k \in E(K)$, of edges such that $s(h)=s(k)$. Then the *edge-sum of H and K (with respect to L_H and L_K)*, denoted $(H, L_H) \oplus_2 (K, L_K)$ or, more commonly, $H \oplus_2 K$, is the graph defined as follows. If neither h nor k is a loop, then $H \oplus_2 K$ is obtained by first identifying h and k head-to-head and tail-to-tail, and then deleting the identified edge. If at least one of h and k is a loop, then $H \oplus_2 K$ is obtained by first contracting h to a vertex v_h and k to a vertex v_k, and then identifying v_h and v_k. We may sometimes refer to $H \oplus_2 K$ as the *edge-sum of H and K along h and k* when L_H and L_K are understood.

It is clear from the definition that edge-summing is commutative. Evidently, if H and K can be edge-summed along h and k (with respect to L_H and L_K), then the edge set of $H \oplus_2 K$ is $(E(H)-h)\dot{\cup}(E(K)-k)$. It is easy to see that there is a partial function $L_{H\oplus_2 K} : E(H \oplus_2 K) \to S \times (V(H \oplus_2 K) \times V(H \oplus_2 K)) : e \mapsto (s(e), (u_{H\oplus_2 K}(e), v_{H\oplus_2 K}(e)))$, where $u_{H\oplus_2 K}(e)$ and $v_{H\oplus_2 K}(e)$ are the vertices in $H \oplus_2 K$ that correspond to the tail and head, respectively, of e determined by L_H or L_K (depending on whether e is in $E(H)-h$ or in $E(K)-k$). Moreover, $L_{H\oplus_2 K}$ is a directed labeling of $H \oplus_2 K$ since $s(e) \neq s(f)$ for any two distinct edges e and f in $(E(H)-h)\dot{\cup}(E(K)-k)$; we shall call $L_{H\oplus_2 K}$ the directed labeling *inherited from L_H and L_K*. If L'_H and L'_K are the directed labelings of, respectively, H and K obtained by reversing the directions assigned by L_H and L_K to the edges h and k, then it is evident that $(H, L_H) \oplus_2 (K, L_K) = (H, L'_H) \oplus_2 (K, L'_K)$. We call this process of obtaining L'_H and L'_K from L_H and L_K *pair direction reversal*. If h is not a block of H, and k is not a block of K, then $H \oplus_2 K$ is called the *2-sum of H and K*.

The following lemma is a well-known property of 2-sums.

Lemma 4.1. *If H and K are 2-connected graphs that can be 2-summed along $h \in E(H)$ and $k \in E(K)$, then $H \oplus_2 K$ is 2-connected.*

We have noted that edge-summing is commutative. In general, edge-summing is not associative, but there is "conditional" associativity. The condition that we must impose is that if H, J, and K are pairwise disjoint graphs with directed labelings L_H, L_J, and L_K, respectively, then exactly two elements of $\{H \oplus_2 J, H \oplus_2 K, J \oplus_2 K\}$ are defined.

Given a collection of pairwise disjoint graphs $\mathscr{G}$ on which we want to perform edge-sums, it is convenient to use a tree T whose vertex set corresponds to $\mathscr{G}$ and whose edge set corresponds to a subset of the set of labels used in the directed labelings of the elements of $\mathscr{G}$. To avoid confusion between vertices and edges of elements of $\mathscr{G}$ and those of T, we shall call elements of $V(T)$ *nodes* and elements of $E(T)$ *links*. Moreover, Greek letters will be used to denote nodes and links of T, and Roman letters will be used to denote vertices and edges of elements of $\mathscr{G}$. We describe this correspondence between $\mathscr{G}$ and T more precisely as follows.

Let $\mathscr{G} = \{G_i\colon i \in [n]\}$ be a collection of pairwise disjoint graphs, let $L_{\mathscr{G}} = \{L_{G_i}\colon i \in [n]\}$ be a collection of directed labelings of the elements of $\mathscr{G}$, and let T be a tree on the node set $\{\xi_i\colon i \in [n]\}$, where n is a nonnegative integer. Then $\mathscr{T} = (\mathscr{G}, L_{\mathscr{G}}, T)$ is an *edge-sum tree* if the following hold:

(i) If $\varepsilon = \xi_i\xi_j \in E(T)$, then there are precisely two graphs of $\mathscr{G}$, namely G_i and G_j, each containing an edge labeled ε.

(ii) If $G_i \in \mathscr{G}$ has an edge labeled ε, then there is exactly one other graph $G_j \in \mathscr{G}$ that has an edge labeled ε; moreover, $\xi_i\xi_j \in E(T)$.

It will be useful to look at a more general kind of tree structure (that includes the edge-sum trees) obtained by relaxing condition (ii). Call $\mathscr{T} = (\mathscr{G}, L_{\mathscr{G}}, T)$ a *labeled edge-sum tree* if and only if $\mathscr{G}$, $L_{\mathscr{G}}$, and T are as above, and $\mathscr{T}$ satisfies condition (i) above and condition (ii)$'$ below.

(ii)$'$ If $G_i \in \mathscr{G}$ has an edge labeled ε, then there is at most one other graph $G_j \in \mathscr{G}$ that has an edge labeled ε, and if there is such a G_j, then $\xi_i\xi_j \in E(T)$.

If $\mathscr{T} = (\mathscr{G}, L_{\mathscr{G}}, T)$ is a labeled edge-sum tree, then call the elements of $\mathscr{G}$ the *node graphs* of $\mathscr{T}$, call $L_{\mathscr{G}}$ the *directed labeling* of $\mathscr{T}$, and call T the *tree* of $\mathscr{T}$.

Given an edge-sum tree $\mathscr{T} = (\mathscr{G}, L_{\mathscr{G}}, T)$ and a subtree T' of T, we can form the edge-sum tree $\mathscr{T}' = (\mathscr{G}', L'_{\mathscr{G}'}, T')$, where $\mathscr{G}'$ is the subcollection of $\mathscr{G}$ corresponding to $V(T')$, by restricting the directed labeling associated with each element of $\mathscr{G}'$ in the appropriate way (that is, for each $G_i \in \mathscr{G}'$, there is an edge of G_i labeled ε if and only if $\varepsilon \in E(T')$ and ξ_i is a vertex of ε). We shall say that $\mathscr{T}'$ is a *restriction* of $\mathscr{T}$ and that $\mathscr{T}'$ is the restriction of $\mathscr{T}$ *induced* by the subtree T' of T. In particular, if the subtree T' is obtained by deleting a leaf ξ from T, then we shall say that $\mathscr{T}'$ is obtained by *deleting* ξ *from* $\mathscr{T}$ and let $\mathscr{T} - \xi$ denote $\mathscr{T}'$.

A basic operation that we shall perform on a labeled edge-sum tree is forming its composition, which we define as follows. Given a labeled edge-sum tree $\mathscr{T} = (\mathscr{G}, L_{\mathscr{G}}, T)$, we can obtain a graph $G(\mathscr{T})$ (with a directed labeling, that is, perhaps, trivial) called the *composition* of $\mathscr{T}$, by edge-summing as dictated by the links of T in the following manner. If T has no links, then T consists of a single node, $\mathscr{G}$ contains exactly one element, namely G_0, and there is nothing to do; hence $G(\mathscr{T})=G_0$, and the edges of $G(\mathscr{T})$ are assigned labels and directions according to L_{G_0}. Inductively, if $E(T)$ is nonempty and $\varepsilon=\xi_i\xi_j$ is a link of T, then form $\mathscr{T}'=(\mathscr{G}', L_{\mathscr{G}'}, T')$, where $\mathscr{G}'$ is obtained from $\mathscr{G}$ by replacing G_i and G_j with their edge-sum, $L_{\mathscr{G}'}$ is obtained from $L_{\mathscr{G}}$ by replacing L_{G_i} and L_{G_j} with the directed labeling $L_{G_i\oplus_2 G_j}$ inherited from L_{G_i} and L_{G_j}, and T' is obtained from T by contracting ε to a node ξ that corresponds to $G_i\oplus_2 G_j$. We say that $\mathscr{T}'$ is obtained from $\mathscr{T}$ by *contracting* ε in $\mathscr{T}$, and let $\mathscr{T}/\varepsilon$ denote $\mathscr{T}'$. It is clear that $\mathscr{T}'$ is a labeled edge-sum tree. In particular, if $\mathscr{T}$ is an edge-sum tree, then so is $\mathscr{T}'$, and it follows that $G(\mathscr{T})$ is unlabeled. In general, when the directed labeling $L_{\mathscr{G}}$ of a labeled edge-sum tree $\mathscr{T}=(\mathscr{G}, L_{\mathscr{G}}, T)$ is understood, we shall let $(\mathscr{G}, T)$ denote $\mathscr{T}$. Also, we shall not indicate when edges of node graphs and compositions are assigned labels and directions except as needed.

It follows from the definition of the composition of a labeled edge-sum tree $\mathscr{T}=(\mathscr{G}, T)$ that there is a sequence $(\mathscr{T}_i)_{i=0}^n$ of labeled edge-sum trees where T has n links, $\mathscr{T}_0=\mathscr{T}$, and $\mathscr{T}_i$ is obtained by contracting a link in $\mathscr{T}_{i-1}$, for each $i\in[n]_+$; it follows that $\mathscr{T}_n=(G(\mathscr{T}), K_1)$. Call each $\mathscr{T}_i$ in the above sequence a *partial composition* of $\mathscr{T}$, and if $i\in[n-1]_+$, then the partial composition $\mathscr{T}_i$ is *proper*. Such a sequence of partial compositions determines a natural way to edge-sum the elements of $\mathscr{G}$.

Fig. 4 shows an edge-sum tree $\mathscr{T}$ and its composition $G(\mathscr{T})$. The nodes of the tree T of $\mathscr{T}$ are indicated by the ovals, and the line segments that connect the ovals are the links of T. Each node graph of $\mathscr{T}$ is drawn inside its corresponding oval. The

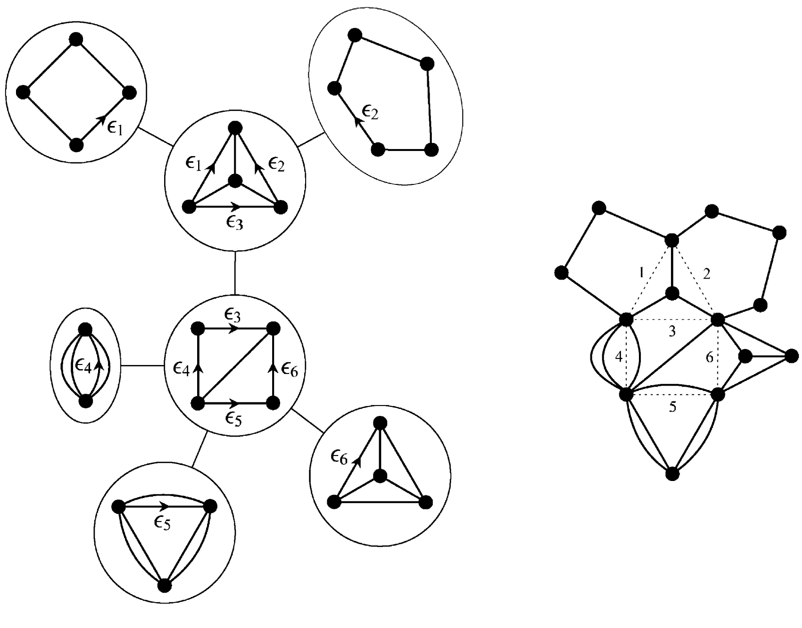

Fig. 4. An edge-sum tree $\mathscr{T}$ and its composition $G(\mathscr{T})$.

directed labeling of $\mathscr{T}$ assigns labels and directions to edges of the node graphs, as indicated. It follows that the line segment that connects the two nodes of T whose node graphs each contain an edge labeled ε_i is the link ε_i. The edges of $G(\mathscr{T})$ are the solid edges. For each $i \in [6]_+$, the dotted line segment labeled i shows where two node graphs were edge-summed along the two edges labeled ε_i (but it is not an edge of $G(\mathscr{T})$).

The terminology has been referring to *the* composition (rather than *a* composition) $G(\mathscr{T})$ of a labeled edge-sum tree $\mathscr{T}$. Indeed, it is routine to verify that any composition of $\mathscr{T}$ results in a unique graph $G(\mathscr{T})$.

Let $\mathscr{T} = (\mathscr{G}, L_{\mathscr{G}}, T)$ and $\mathscr{T}' = (\mathscr{G}, L'_{\mathscr{G}}, T)$ be directed edge-sum trees such that $L'_{\mathscr{G}}$ is obtained from $L_{\mathscr{G}}$ by a sequence of pair direction reversals. We say that $\mathscr{T}$ and $\mathscr{T}'$ are *equivalent*. Indeed, it is easy to see that $G(\mathscr{T})$ and $G(\mathscr{T}')$ are the same.

If each element of $\mathscr{G}$ is 2-connected, then an edge-sum tree $\mathscr{T} = (\mathscr{G}, T)$ is a *block tree*. The next important kind of edge-sum tree, namely 3-block tree, due to Tutte, requires the following terminology. A 3-*block* is a simple 3-connected graph, a cycle with at least 3 edges, or a multi-edge with at least 3 edges. A 3-*block tree* is an edge-sum tree $\mathscr{T} = (\mathscr{G}, T)$ such that each element of $\mathscr{G}$ is a 3-block and such that if $\xi_i\xi_j \in E(T)$, then G_i and G_j are not both cycles and not both multi-edges.

Obviously, a 3-block tree is a block tree. Let us note that the edge-sum tree $\mathscr{T}$ that we saw in Fig. 4 is a block tree, but not a 3-block tree. It follows easily from the above proposition and Lemma 4.1 that composing a block tree produces a unique (unlabeled) 2-connected graph. It is natural to ask whether every 2-connected graph has a decomposition into some kind of block tree. Indeed, Tutte [7] proved the following:

Theorem 4.2. *If G is a* 2-*connected graph containing at least three edges, then it can be decomposed into a* 3-*block tree. Moreover, this decomposition is unique* (*up to equivalence of* 3-*block trees*).

Later, we shall use the existence of such a decomposition guaranteed by Theorem 4.2.

For brevity, let us speak of the 3-block tree of a 2-connected graph G rather than the class of equivalent 3-block trees of G. Next, we shall prove a useful lemma regarding the composition of a special kind of restriction of an edge-sum tree.

Lemma 4.3. *If $\mathscr{T} = (\mathscr{G}, T)$ is an edge-sum tree and $\mathscr{T}' = (\mathscr{G}', T')$ is a restriction of $\mathscr{T}$ so that, for each node ξ_j in $V(T) - V(T')$, the corresponding node graph G_j is* 2-*connected, then $G(\mathscr{T}') \leqslant_{\mathrm{m}} G(\mathscr{T})$.*

Proof. We show that the hypotheses imply a stronger conclusion, namely $G(\mathscr{T}') \leqslant_{\mathrm{t}} G(\mathscr{T})$. We may assume that $\mathscr{T}' = \mathscr{T} - \xi_j$, where ξ_j is a leaf of T whose corresponding node graph G_j is 2-connected, since any subtree T' can be obtained from T by deleting leaves and since the taking of restrictions of edge-sum trees and the $\leqslant_{\mathrm{t}}$ relation on graphs are transitive. Let $\varepsilon = \xi_i\xi_j$ denote the link of T incident with ξ_j. Then

$(\{H, G_j\}, \varepsilon)$ is a partial composition of $\mathscr{T}$, where H, viewed as an unlabeled graph, is isomorphic to $G(\mathscr{T}')$. In this partial composition, each of H and G_j has an edge, respectively, h and g, both of which are labeled ε. Since G_j is 2-connected, there is a cycle C of length at least 2 that contains g. It follows that $H \oplus_2 C \leqslant_{\mathrm{s}} H \oplus_2 G_j = G(\mathscr{T})$. Note that $H \oplus_2 C$ is isomorphic to the unlabeled graph obtained from H by subdividing h with $|C| - 2$ new vertices. Hence, $H \leqslant_{\mathrm{t}} H \oplus_2 C$. Since $H \cong G(\mathscr{T}')$, it follows that $G(\mathscr{T}') \leqslant_{\mathrm{t}} G(\mathscr{T})$. □

The following is an immediate and useful corollary of Lemma 4.3.

Corollary 4.4. *Let G be a 2-connected graph, and let $\mathscr{T} = (\mathscr{G}, T)$ be its 3-block tree. If some element of $\mathscr{G}$ is a 3-connected graph that has a cycle of length at least N, where N is the number from Theorem 3.7, then $W_n \leqslant_{\mathrm{m}} G$ (and hence $F_n \leqslant_{\mathrm{m}} G$).*

We shall need the following two well-known binary relations on graphs, which are more permissive versions of isomorphism. A graph G is 2-*isomorphic* to a graph H, denoted $G \cong_2 H$, if there is a positive integer n and a sequence $(G_i)_{i=1}^{n}$ of graphs such that $G_1 = G$, the final graph $G_n = H$, and if $i \in [n-1]_+$, then G_{i+1} is obtained by performing one of the three following operations on G_i:

(i) *Vertex identification*: If v_1 and v_2 are vertices in distinct components of G_i, then G_{i+1} is obtained by identifying v_1 and v_2 to a new vertex v.
(ii) *Vertex cleaving*: If G^1 and G^2 are disjoint graphs such that G_i can be obtained from G^1 and G^2 by identifying a vertex v_1 of G^1 and a vertex v_2 of G^2 to a single vertex v, then let $G_{i+1} = G^1 \cup G^2$.
(iii) *Twisting*: Assume that G^1 and G^2 are disjoint graphs and that u_1, u_2, v_1, and v_2 are all distinct vertices with $\{u_1, v_1\} \subseteq V(G^1)$ and $\{u_2, v_2\} \subseteq V(G^2)$. Further, assume that G_i is obtained from G^1 and G^2 by identifying u_1 and u_2 to a single vertex u and by identifying v_1 and v_2 to a single vertex v. Call G_{i+1} a *twisting of G_i about $\{u, v\}$* if G_{i+1} is obtained from G^1 and G^2 by identifying u_1 and v_2 to a single vertex u', and by identifying u_2 and v_1 to a single vertex v'.

If, in the process of obtaining H from G, only operations (i) and (ii) are used, we say that G is 1-*isomorphic* to H and we write $G \cong_1 H$. Note that if H can be obtained from G by adding isolated vertices, then $G \cong_1 H$.

The following lemma is straightforward—we leave its proof to the reader.

Lemma 4.5. *If $\mathscr{T} = (\mathscr{G}, L_{\mathscr{G}}, T)$ is an edge-sum tree, and $\mathscr{T}' = (\mathscr{G}, L'_{\mathscr{G}}, T)$ is an edge-sum tree obtained from $\mathscr{T}$ by reversing the directions of some of the labeled edges of elements of $\mathscr{G}$, then $G(\mathscr{T})$ is 2-isomorphic to $G(\mathscr{T}')$.*

Let $\mathscr{T} = (\mathscr{G}, L_{\mathscr{G}}, T)$ be an edge-sum tree for which $|\mathscr{G}| > 1$, let H be a specified node graph in $\mathscr{G}$, and let ξ be the node that corresponds to H. Let the positive integer m denote the number of links in T incident with ξ, and let $\{\varepsilon_i \colon i \in [m]_+\}$ denote the set of links adjacent to ξ in T. Then the *star of* $\mathscr{T}$ (*at* H), denoted $\mathscr{T}_*$, is the partial composition $\mathscr{T}/(E(T)-\{\varepsilon_i\colon i\in[m]_+\})$ of $\mathscr{T}$. We now define some additional notation regarding $\mathscr{T}$ and $\mathscr{T}_*$. For each $i\in[m]_+$, let h_i be the edge of H that is labeled ε_i, let ξ_i be the endnode of ε_i in T that is not ξ, let H_i be the node graph of $\mathscr{T}$ corresponding to ξ_i, and let k_i be the edge of H_i that is labeled ε_i. Let $\mathscr{T}_i$ be the restriction of $\mathscr{T}$ induced by the component T_i of $T\backslash\varepsilon_i$ containing ξ_i, and let $K^i=G(\mathscr{T}_i)$.

It is straightforward that the set of node graphs of $\mathscr{T}_*$ is $\{H\}\cup\{K^i\colon i\in[m]_+\}$, where H is labeled as it is in $\mathscr{T}$, and where K^i has exactly one labeled edge, namely k_i, for each $i\in[m]_+$.

The next lemma states that the operations of edge deletion and edge contraction commute with the process of forming the edge-sum tree. The proof is straightforward and its details are left for the reader.

Lemma 4.6. *Let* $\mathscr{T}=(\mathscr{G},L_{\mathscr{G}},T)$ *be an edge-sum tree, let* D *and* C *be disjoint subsets of* $E(G(\mathscr{T}))$, *and let* $\mathscr{T}\backslash D/C$ *denote the edge-sum tree obtained by replacing each node graph* $H\in\mathscr{G}$ *with* $H'=H\backslash(E(H)\cap D)/(E(H)\cap C)$, *and by replacing each directed labeling* L_H *in* $L_{\mathscr{G}}$ *with the directed labeling* $L_{H'}$ *of* H' *induced by* L_H. *Then* $G(\mathscr{T}\backslash D/C)=G(\mathscr{T})\backslash D/C$.

5. A long path in a 3-block tree

The following is the main result of this section.

Theorem 5.1. *Let* G *be a* 2-*connected graph with at least three edges, and let* $\mathscr{T}=(\mathscr{G},T)$ *be its* 3-*block tree. If* n *is a positive integer, and* T *contains a path of length at least* $4(n-1)+1$ *as a subgraph, then* $F_n\leqslant_{\mathrm{m}}G$.

Before proving Theorem 5.1, we need an auxiliary result, which is stated as Corollary 5.3 below. We begin by stating a result of Seymour [5].

Theorem 5.2. *If* M *is a* 3-*connected matroid that has a minor in the set* $\mathscr{F}=\{U_{2,4},M(K_4)\}$, *and* X *is any subset of* $E(M)$ *that has at most two elements, then* M *has a minor in* $\mathscr{F}$ *using* X.

Two well-known facts: that every 3-connected graph contains a minor isomorphic to K_4 and that the matroid $U_{2,4}$ is not graphic, together with Theorem 5.2, immediately imply the following:

Corollary 5.3. *If* G *is a simple* 3-*connected graph, and* e *and* f *are edges of* G, *then there is a* K_4-*minor of* G *that uses* e *and* f.

The remainder of this section is devoted to proving Theorem 5.1.

Proof of Theorem 5.1. Assume that T contains a subtree P_0 that is a path of length at least $4(n-1)+1$. If each of the elements of $\mathscr{G}$ corresponding to the endnodes of P_0 is a multi-edge, then let P be a subpath of P_0 obtained by deleting an endnode from P_0; otherwise let $P=P_0$. Then T contains a subpath P of length N, for some integer $N \geqslant 4(n-1)$, one endnode of which corresponds to a 3-connected graph or a cycle. Let $\mathscr{T}'=(\mathscr{G}',P)$ be the 3-block tree that is the restriction of $\mathscr{T}$ induced by P, and let $G'=G(\mathscr{T}')$. By Lemma 4.3, $G' \leqslant_{\mathrm{m}} G$.

By renumbering indices, we may assume that the node set of P is $\{\xi_i\colon i\in[N]\}$, the link set of P is $\{\varepsilon_i=\xi_{i-1}\xi_i\colon i\in[N]_+\}$, and $\mathscr{G}'=\{G_i\colon i\in[N]\}$, where G_i is the 3-block corresponding to ξ_i, and G_N is not a multi-edge. We now want to partition $\mathscr{G}'$ into singletons and pairs as follows. Let i be the largest index such that G_i does not belong to a singleton or a pair of elements of $\mathscr{G}'$. If G_{i-1} is a multi-edge, then form the pair $\{G_{i-1},G_i\}$; otherwise, form the singleton $\{G_i\}$. If all elements of $\mathscr{G}'$ have not been placed in a singleton or a pair, then repeat this process. It is straightforward that this process produces a partition $\mathscr{P}(\mathscr{G}')$ of $\mathscr{G}'$ where each element of $\mathscr{P}(\mathscr{G}')$ is a singleton consisting of a cycle or a 3-connected graph, or a pair $\{G_{i-1},G_i\}$ consisting of a multi-edge G_{i-1} and a 3-block G_i that is not a multi-edge. Since each element of $\mathscr{P}(\mathscr{G}')$ consists of at most two 3-blocks, $|\mathscr{P}(\mathscr{G}')|=N'+1$ for some integer $N' \geqslant \lceil (N+1)/2\rceil-1$. Note that it follows from the way that $\mathscr{P}(\mathscr{G}')$ is defined and from the fact that $\mathscr{G}'$ is a 3-block tree that if $i\in[N]_+$, and G_i is a cycle that makes up a singleton in $\mathscr{P}(\mathscr{G}')$, then G_{i-1} is 3-connected.

Note that $E(P)$ is partitioned into sets E' and E'' of links such that if $\varepsilon'\in E'$, then the node graphs that contain an edge labeled ε' are contained in different elements of $\mathscr{P}(\mathscr{G}')$, and if $\varepsilon''\in E''$, then the node graphs that contain an edge labeled ε'' form a pair in $\mathscr{P}(\mathscr{G}')$.

Let $\mathscr{T}''$ be the block tree obtained by contracting E'' in $\mathscr{T}'$. It follows that $P'=P/E''$ is the tree of $\mathscr{T}''$, and $\mathscr{G}''=\{G_i\colon \{G_i\}\in\mathscr{P}(\mathscr{G}')\}\cup\{G_i\oplus_2 G_{i+1}\colon \{G_i,G_{i+1}\}\in\mathscr{P}(\mathscr{G}')\}$ is the set of node graphs of $\mathscr{T}''$. Furthermore, $|\mathscr{G}''|=|\mathscr{P}(\mathscr{G}')|=N'+1$. Also, it is evident that $G(\mathscr{T}'')=G(\mathscr{T}')=G'$ since $\mathscr{T}''$ is a partial composition of $\mathscr{T}'$. Let G'_0 denote the element of $\mathscr{G}''$ that is either G_N or $G_{N-1}\oplus_2 G_N$, let ξ'_0 denote the endnode of P' corresponding to G'_0, and if $E(P')$ is nonempty, then let ε'_1 denote the link of P' incident with ξ'_0. If P' contains additional links, then let $\{\varepsilon'_i\}_{i=2}^{N'}$ denote the remaining links of P' such that ε'_i and ε'_{i+1} are adjacent for each $i\in[N'-1]_+$, and rename the nodes of P' and elements of $\mathscr{G}''$ such that, for each $i\in[N']_+$, the endnodes of the link ε'_i are ξ'_{i-1} and ξ'_i, and G'_i is the element of $\mathscr{G}''$ corresponding to ξ'_i. It follows that if G'_{i-1} is a cycle, then G'_i is a 3-connected graph.

Note that each element of $\mathscr{G}''$ is a 3-block that is not a multi-edge, or it is the 2-sum of a 3-block that is a multi-edge and a 3-block that is not a multi-edge. It follows that all edges of G'_0 are unlabeled except for one edge f_0 that is labeled ε'_1, and all edges of $G'_{N'}$ are unlabeled except for one edge $e_{N'}$ that is labeled $\varepsilon'_{N'}$. Furthermore, if G'_0 is not a simple graph, then f_0 is contained in the proper multi-edge of G'_0. Also, if $G'_{N'}$ is not a simple graph, then $e_{N'}$ is a trivial multi-edge of $G'_{N'}$. If $i\in[N'-1]_+$, then all edges of G'_i are unlabeled except for an edge e_i that is labeled ε'_i and an edge f_i that

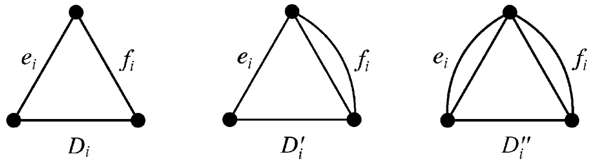

Fig. 5. The graphs D_i, D_i', and D_i''.

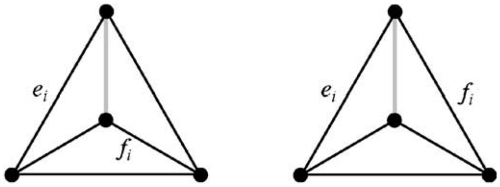

Fig. 6. One of the above graphs is a minor of G_i'.

is labeled ε_{i+1}'. Moreover, if G_i' is not simple, then e_i is a trivial multi-edge of G_i', and f_i belongs to the proper multi-edge of G_i'. Let e_0 be an unlabeled trivial multi-edge of G_0', and let $f_{N'}$ be an edge that is contained in a largest unlabeled multi-edge of $G_{N'}'$, so that each element G_i' of $\mathscr{G}''$ has exactly two specified edges e_i and f_i.

We shall show that, for each $i \in [N']$, the graph G_i' contains a particular minor isomorphic to one the three graphs in Fig. 5. First, we shall show that if G_i' is a 3-block that is a cycle, then $D_i \leqslant_{\mathrm{m}} G_i'$. Then, we shall show that if G_i' is the 2-sum of a 3-block that is a cycle and a 3-block that is a multi-edge, then $D_i' \leqslant_{\mathrm{m}} G_i'$. For the remaining case, in which G_i' is a 3-connected graph, we shall show that $D_i'' \leqslant_{\mathrm{m}} G_i'$.

First, assume that G_i' is a 3-block that is a cycle. Since G_i' has at least three edges, $G_i' \backslash \{e_i, f_i\}$ consists of a proper path P_1 and a (perhaps trivial) path P_2. By contracting, in G_i', the paths P_1 to a single edge and P_2 to a vertex, we obtain a graph G_i'' that is isomorphic to D_i.

Now, assume that G_i' is the 2-sum of a 3-block C that is a cycle and a 3-block C^* that is a multi-edge. Clearly, the simplification of G_i' is a cycle with at least three edges. As already mentioned, e_i is a trivial multi-edge of G_i', and f_i is contained in the proper multi-edge of G_i'. As in the case in which G_i' is a cycle, the graph obtained by deleting the proper multi-edge and e_i from G_i' consists of a proper path P_1 and a (perhaps trivial) path P_2. If we contract, in G_i', the paths P_1 to a single edge and P_2 to a vertex, then the resulting graph contains a subgraph G_i'' that is isomorphic to D_i'.

Finally, assume that G_i' is a 3-connected graph. If G_i' is not simple, then e_i is a trivial multi-edge, and f_i is contained in the proper multi-edge of G_i', and hence, e_i and f_i are not parallel. Consequently, if G_i' is not simple, then we may take the simplification $\widetilde{G_i'}$ of G_i' so that $\{e_i, f_i\} \subseteq E(\widetilde{G_i'})$. By Corollary 5.3, $\widetilde{G_i'}$ has a K_4-minor using e_i and f_i. Thus, one of the two graphs in Fig. 6 is a minor of $\widetilde{G_i'}$, and, by contracting the shaded edge in either graph, we obtain a graph G_i'' that is isomorphic to D_i''. Since $\widetilde{G_i'} \leqslant_{\mathrm{s}} G_i'$, it is clear that $G_i'' \cong D_i'' \leqslant_{\mathrm{m}} G_i'$.

Let $\mathscr{T}''' = (\mathscr{G}''', P')$ be the block tree where $\mathscr{G}''' = \{G_i'' : i \in [N']\}$ and G_i'' is the graph that corresponds to the node ξ_i' of P'. Since, for each $i \in [N']$, the graph G_i'' is a minor of G_i' in which no labeled edges are contracted, it follows from Lemma 4.6 that the composition G'' of $\mathscr{T}'''$ is a minor of G'. Moreover, it follows from Lemma 4.1 that G'' is 2-connected (hence, loop-free). Next, we want to show that G'' is a graph that is, in some sense, similar to a fan.

So far, we have been disregarding the directions assigned by the directed labeling of $\mathscr{T}'''$ to the labeled edges in the elements of $\mathscr{G}'''$. We consider these directions now. By performing the appropriate pair direction reversals, we may assume that, for each $i \in [N'-1]$, the edge f_i of G_i'' is directed so that its head is incident with e_i. Let $\mathscr{T}^*$ denote the block tree obtained from $\mathscr{T}'''$ by directing $e_i \in E(G_i'')$ so that its head is incident with f_i, for each $i \in [N']_+$. Since the simplification of each G_i'' is a triangle (that is, a 3-cycle) for each $i \in [N']$, call the vertex common to e_i and f_i the *point* of G_i'', and let g_i denote the edge of G_i'' that is not adjacent to the point of G_i''.

We want to show that the simplification of G'' is 2-isomorphic to $F_{N'+2}$. By Lemma 4.5 it suffices to show that the simplification of $G(\mathscr{T}^*)$ is isomorphic to $F_{N'+2}$. Informally, in the composition of $G(\mathscr{T}^*)$, the first node graph G_0'' contributes 2 to the size of the fan, and each additional node graph G_i'' contributes 1 to the size of the fan. Let us recall that if G_{i-1}' is a cycle, then G_i' is a 3-connected graph. It follows that if $G_{i-1}'' \cong D_{i-1}$, then $G_i'' \cong D_i''$. It is straightforward that $\widetilde{G(\mathscr{T}^*)} \cong F_{N'+2}$, given the way that the labeled edges of $\mathscr{T}^*$ are directed and the fact that if $G_{i-1}'' \cong D_{i-1}$, then $G_i'' \cong D_i''$. Hence, $\widetilde{G''} \cong_2 F_{N'+2}$.

Next, we want to show that the $\lceil (N'+2)/2 \rceil$-fan is a minor of $\widetilde{G''}$. Since $\widetilde{G''}$ is 2-isomorphic to $F_{N'+2}$ and 2-connected, $\widetilde{G''}$ can be obtained from a finite sequence of twistings of $F_{N'+2}$ about vertex-cuts of size two. It is straightforward that $\widetilde{G''}$ is similar to a fan, where some of the triangles may point up and some may point down instead of all triangles pointing in the same direction.

Define the function $f : \mathscr{G}''' \to \{-1, 1\}$ as follows. Let $f(G_0'') = 1$, and inductively, for each $i \in [N']_+$, if the directed labeling of $\mathscr{T}'''$ directs e_i so that its head is the point of G_i'', then $f(G_i'') = f(G_{i-1}'')$; otherwise, $f(G_i'') = -f(G_{i-1}'')$. Informally, we shall say that the triangle of $\widetilde{G''}$ with base g_i *points up* if $f(G_i'') = 1$ and *points down* if $f(G_i'') = -1$. It follows that if $\sum_{i=0}^{N'} f(G_i'') \geqslant 0$, then at least half of the triangles of $\widetilde{G''}$ point up; otherwise, more than half of the triangles point down. If $\sum_{i=0}^{N'} f(G_i'') \geqslant 0$, then contract $\{g_i : f(G_i'') = -1\}$ in $\widetilde{G''}$; otherwise, contract $\{g_i : f(G_i'') = 1\}$. It follows that the simplification of the resulting graph is isomorphic to a fan of size at least $\lceil (N'+2)/2 \rceil$. Hence, $F_{\lceil (N'+2)/2 \rceil} \leqslant_{\mathrm{m}} \widetilde{G''} \leqslant_{\mathrm{m}} G'$, and

$$\left\lceil \frac{N'+2}{2} \right\rceil \geqslant \left\lceil \frac{\lceil \frac{N+1}{2} \rceil + 1}{2} \right\rceil \geqslant \left\lceil \frac{\lceil (4(n-1)+1)/2 \rceil + 1}{2} \right\rceil = \left\lceil \frac{2(n-1)+2}{2} \right\rceil = n.$$

Thus, $F_n \leqslant_{\mathrm{m}} G$. $\square$

6. Comulticycles and multipaths

In this section we state and prove a lemma which states that if a graph G satisfies certain conditions that depend, in part, on an integer n exceeding 3, then an element of $\{C^*_{n,n}, P_{n,n}\}$ is a minor of G, where $P_{n,n}$ is obtained from the path P_n on n edges by replacing each edge of P_n with a multi-edge of size n. Following the proof of the lemma, we state two corollaries, which describe the consequences of the lemma to 2-connected graphs and block trees.

Lemma 6.1. *Let G be a graph with two specified vertices x and y such that $G \cup e$ is 2-connected, where $e \| xy$, and let n be an integer exceeding 3. If every xy-path in G has length at least $n(n-1)$ and every xy-edge-cut in G has size at least $n2^{n^3}$, then at least one of the following holds*:

(i) $C^*_{n,n} \leqslant_{\mathrm{m}} G$, *and the vertices of $C^*_{n,n}$ that have degree n are x and y.*
(ii) $P_{n,n} \leqslant_{\mathrm{m}} G$, *and the endvertices of $P_{n,n}$ are x and y.*

Proof. Label each vertex v of G with its distance $l(v)$ from x. Then $l(y) = N$ for some $N \geqslant n(n-1)$. For $i \in [N]$, let V_i be the set of those vertices v labeled with i such that there is a vy-path in G each of whose vertices, except v, is labeled with an integer exceeding i. It is clear that V_i is nonempty if $i \in [N]$, that $V_0 = \{x\}$ and $V_N = \{y\}$, and that V_i is an xy-vertex-cut if $i \in [N-1]_+$. Let V_i^x be the set of vertices in the component of $G - V_i$ containing x for $i \in [N]_+$, and let V_i^y be the set of vertices in the component of $G - V_i$ containing y for $i \in [N-1]$. We now establish some properties of these two sets.

(1) $V_{i'} \subseteq V_i^x$ if $0 \leqslant i' < i \leqslant N$.

To see this, it suffices to show that, for every $v \in V_{i'}$, there is an xv-path in $G - V_i$. Let v be an arbitrarily chosen vertex in $V_{i'}$. Since the label on v is determined by its distance from x, it follows that G contains an xv-path P_x of length $l(v) = i'$ and that the label of each vertex of P_x is at most i'. In particular, P_x has no vertex of V_i, and thus P_x is contained in $G - V_i$, as required.

(2) $V_{i'} \subseteq V_i^y$ if $0 \leqslant i < i' \leqslant N$.

The proof of (2) is very similar to the proof of (1). Let v be an element of $V_{i'}$. It follows that G contains a vy-path P_y and that the label of each vertex of P_y is at least i'. In particular, P_y contains no vertex of V_i, and thus P_y is contained in $G - V_i$. Consequently, (2) holds.

For each i in $[N-n]$, define $V_i^0 = V_i^y \cap V_{i+n}^x$. Statements (1) and (2) immediately imply that for each such i, $V_{i+1} \subseteq V_i^0$, and hence V_i^0 contains an xy-vertex-cut.

Assume first that there is an $i \in [N-n]$ such that a smallest xy-vertex-cut S_i of G contained in V_i^0 has at least n vertices. Since G is connected, it is clear that there are three kinds of bridges of $V_i \cup V_{i+n}$ in G: those that meet only V_i, those that meet only V_{i+n}, and those that meet both V_i and V_{i+n}.

Now, we want to contract all of the bridges of $V_i \cup V_{i+n}$ except those that meet both V_i and V_{i+n}. More precisely, let us contract $E_0 = E(G) - (E(G[V_i^0]) \cup E(V_i, V_i^0) \cup E(V_i^0, V_{i+n}))$ in G, where $E(X_1, X_2)$ denotes the set of edges whose elements have one vertex in X_1 and the other vertex in X_2 for disjoint sets X_1 and X_2 of vertices. Let $G_0 = G/E_0$. On contracting E_0 in G, it is easy to see that x and the vertices of V_i are identified, and that y and the vertices of V_{i+n} are identified. It is natural to let x and y, respectively, denote these vertex identifications. Consequently, $V(G_0) = V_i^0 \cup \{x, y\}$.

The next part of the proof uses the following two simple observations. First, no edge of E_0 has a vertex in V_i^0. Second, two vertices are in the same component of a graph if and only if after contracting any set of edges, those vertices (which may become identified) are in the same component of the resulting graph.

We now show that S_i is a smallest xy-vertex-cut of G_0 by showing that S_i contains an xy-vertex-cut of G_0, and then showing that no subset of $V(G_0) - \{x, y\}$ having size less than $|S_i|$ is an xy-vertex-cut of G_0. Clearly x and y are in different components of $G - S_i$ since S_i is an xy-vertex-cut of G. Also, in view of the first observation above, $(G - S_i)/E_0$ is well defined since $S_i \subseteq V_i^0$. Thus, $(G - S_i)/E_0 = (G/E_0) - S_i = G_0 - S_i$, and hence x and y are in different components of $G_0 - S_i$ by the second observation above. Now let S be any subset of V_i^0 that has fewer than $|S_i|$ vertices. Then x and y are in the same component of $G - S$ since S_i is a smallest xy-vertex-cut of G contained in V_i^0. By the first observation above, $(G - S)/E_0$ is well defined since $S \subseteq V_i^0$, and thus $(G - S)/E_0 = G_0 - S$. By the second observation above, x and y are in the same component of $G_0 - S$. Hence, no subset of V_i^0 that has fewer than $|S_i|$ vertices is an xy-vertex-cut of G_0. It follows that S_i is a smallest xy-vertex-cut of G_0. Since $|S_i| \geqslant n$, (1) implies that there are xy-paths $P_1, P_2, \ldots, P_n$ in G_0 that are pairwise internally vertex-disjoint.

Now, we show that each P_j has length at least n for $j \in [n]_+$. It follows from the first observation above that $G[V_i^0] = G_0[V_i^0]$. Let P be any xy-path in G_0. Then $P' = P - \{x, y\}$ is a path in $G_0[V_i^0] = G[V_i^0]$. Hence, P' is a path in G that has one endvertex adjacent to some vertex of V_i and the other endvertex adjacent to some vertex of V_{i+n}. It is clear that if two vertices are adjacent in G, then their labels differ by 0 or 1. This implies that if the labels of the endvertices of a path in G are l_1 and l_2, then the length of that path is at least $|l_2 - l_1|$. Furthermore, one endvertex of P' is labeled at most $i + 1$, and the other endvertex is labeled at least $i + n - 1$. So the length of P' is at least $(i + n - 1) - (i + 1) = n - 2$. Hence, the length of P in G_0 is at least n. In particular, P_j has length at least n, for each $j \in [n]_+$.

Let G_0' be the subgraph of G_0 that is the union of $P_1, P_2, \ldots, P_n$. Then G_0' consists of n pairwise internally vertex-disjoint xy-paths, all of length at least n. On contracting an appropriate number of interior edges of P_j in G_0', for each $j \in [n]_+$, we obtain a minor of G_0' that is isomorphic to $C_{n,n}^*$, whose vertices of degree n are x and y. So $C_{n,n}^* \leqslant_{\mathrm{m}} G_0' \leqslant_{\mathrm{s}} G_0 \leqslant_{\mathrm{m}} G$ (hence, $C_{n,n}^* \leqslant_{\mathrm{m}} G$), and the vertices of degree n of $C_{n,n}^*$ are x and y. Thus, the lemma holds if there is an $i \in [N - n]$ such that V_i^0 lacks an xy-vertex-cut of size less than n.

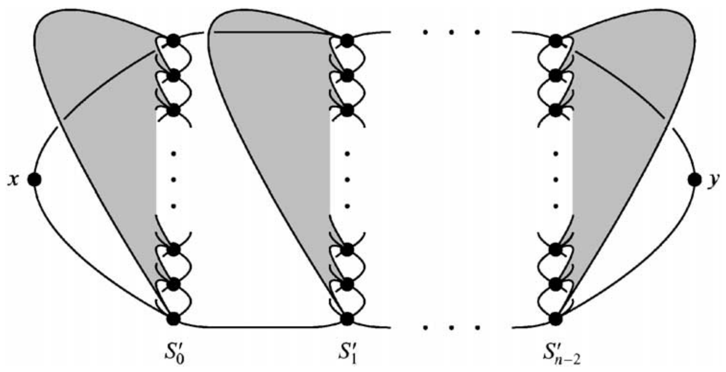

Fig. 7. The structure of the bridges of $\bigcup_{i=0}^{n-2} S'_i$ in G.

Now, for the remaining case, assume that, for each $i \in [N-n]$, if S_i is a smallest xy-vertex-cut in V_i^0, then $|S_i| < n$. Let S'_i be a smallest xy-vertex-cut from V_{in}^0 for each $i \in [n-2]$. As S'_i is an xy-vertex-cut, each xy-path must pass through some vertex s_i in S'_i, for each $i \in [n-2]$.

Let us consider the bridges of $\bigcup_{i=0}^{n-2} S'_i$ in G. Since G is connected, possibly, we could have the following kinds of bridges: those that meet exactly one S'_i, those that meet only S'_i and S'_{i+1} for some $i \in [n-3]$, and those that meet S'_i and S'_j (and, perhaps, additional sets S'_k) for some $0 \leqslant i < j-1 < n-2$. Next, we show that G has no bridges of the last kind by showing that any $s_i s_j$-path in G contains a vertex $s_{i+1} \in S'_{i+1}$, when $0 \leqslant i < j-1 < n-2$, $s_i \in S'_i$, and $s_j \in S'_j$.

First, we point out that if the labels of the endvertices of a path in G are l_1 and l_2, then certainly the path has at least one vertex labeled l' for each integer l' between l_1 and l_2, since the labels of adjacent vertices in G differ by 0 or 1. It follows that if P is an $s_i s_j$-path in G, where $0 \leqslant i < j-1 < n-2$, and s_i and s_j are arbitrary elements of S'_i and S'_j, respectively, then P contains a vertex whose label is $(i+1)n$ since $l(s_i) = in$ and $l(s_j) = jn$. Let s_{i+1} be the vertex labeled $(i+1)n$ that is closest in P to s_j. Then each vertex of the $s_{i+1}s_j$-subpath of P, except s_{i+1}, is labeled greater than $(i+1)n$. Since $s_j \in S'_j$, there is an $s_j y$-path each of whose vertices is labeled at least jn. The union of the $s_{i+1}s_j$-subpath and the $s_j y$-path contains an $s_{i+1}y$-path each of whose vertices is labeled greater than $(i+1)n$, except s_{i+1}, which is labeled $(i+1)n$. Hence, $s_{i+1} \in S'_{i+1}$, which establishes that G has no bridges that meet S'_i and S'_j, where $0 \leqslant i < j-1 < n-2$. Consequently, the structure of the bridges of $\bigcup_{i=0}^{n-2} S'_i$ in G is as in Fig. 7.

Now, let us consider the minor G_1 of G that is obtained by contracting those bridges of $\bigcup_{i=0}^{n-2} S'_i$ in G that contain neither x nor y and that meet only S'_i, for each $i \in [n-2]$. These bridges are represented by the shaded portions of G in Fig. 7. We note that, for each $i \in [n-2]$, some vertices of S'_i may become identified on contracting G to G_1; let $S'_{i,1}$ denote the subset of $V(G_1)$ that corresponds to $S'_i \in V(G)$. It is clear that $|S'_{i,1}| \leqslant |S'_i|$.

We now consider the minor G_2 of G_1 that is obtained by contracting the edge set E_1 contained in G_1 that is defined as follows. $E_1 = E(G_1) - (E(\{x\}; y) \cup \bigcup_{i=0}^{n-2} E(S'_{i,1}; y))$,

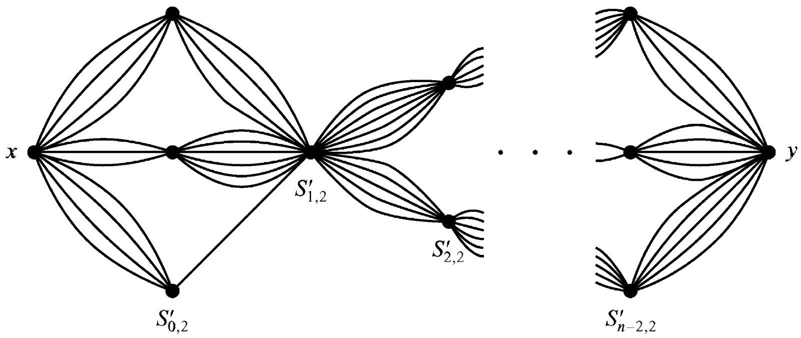

Fig. 8. A typical G_2.

where $E(S; y)$ is the set of those edges of G_1 each of which has one vertex in S and the other vertex in the component of $G_1 - S$ containing y, for $S \subseteq V(G_1) - y$. For each $i \in [n-2]$, some vertices of $S'_{i,1}$ may become identified on contracting G_1 to G_2; let $S'_{i,2}$ denote the set of vertices of G_2 that corresponds to $S'_{i,1}$ in G_1. Then $|S'_{i,2}| \leqslant |S'_{i,1}|$, and $G_2 = G_1/E_1$. Fig. 8 shows a typical G_2.

We now show that G_2 has at least $n2^{n^3}$ pairwise edge-disjoint xy-paths. Recall that every xy-edge-cut of G has size at least $n2^{n^3}$. Then by the well-known Menger's Theorem, G has at least $n2^{n^3}$ pairwise edge-disjoint xy-paths. Note that, given any xy-path P of G, if we contract (in G) a set S of edges that contains no xy-path, then the subgraph P' of G/S induced by $E(P) - S$ is connected, and hence P' contains an xy-path P''. Moreover, $E(P'') \subseteq E(P') \subseteq E(P)$. This containment and the fact that G has at least $n2^{n^3}$ pairwise edge-disjoint xy-paths imply that G_2 has at least $n2^{n^3}$ pairwise edge-disjoint xy-paths. Next, we show that the simplification of G_2 has fewer than 2^{n^3} xy-paths.

Note that each edge of G_2 is of the form xs_0, $s_{n-2}y$, or s_is_{i+1}, where $s_0 \in S'_{0,2}$, $s_{n-2} \in S'_{n-2,2}$, and $s_i \in S'_{i,2}$, for $i \in [n-3]$. It follows that each xy-path in G_2 has length at least n. Moreover, the simplification $\widetilde{G_2}$ of G_2 has at most $|S'_{0,2}|$ edges between x and $S'_{0,2}$, at most $|S'_{i,2}||S'_{i+1,2}|$ edges between $S'_{i,2}$ and $S'_{i+1,2}$ if $i \in [n-3]$, and at most $|S'_{n-2,2}|$ edges between $S'_{n-2,2}$ and y. Since $|S'_{i,2}| \leqslant |S'_i| < n$, for each $i \in [n-2]$, $\widetilde{G_2}$ has at most $n-1+(n-2)(n-1)^2+n-1$ edges, and hence $\widetilde{G_2}$ has fewer than n^3 edges. Clearly, the collection of xy-paths in $\widetilde{G_2}$ is contained in the collection $\mathscr{G}$ of subgraphs of $\widetilde{G_2}$ that lack isolated vertices. Since $|\mathscr{G}| < 2^{n^3}$, there are fewer than 2^{n^3} xy-paths in $\widetilde{G_2}$.

Since G_2 has at least $n2^{n^3}$ pairwise edge-disjoint xy-paths and $\widetilde{G_2}$ has fewer than 2^{n^3} xy-paths, there are at least n pairwise edge-disjoint xy-paths, $P'_1, P'_2, \ldots, P'_n$ in G_2, each of length at least n, that use the same vertices in the same order. If the length of P'_j is greater than n, for each j in $[n]_+$, then we can contract in $\bigcup_{j=1}^n P'_j$ a parallel class whose edges are incident to neither x nor y repeatedly until we obtain a graph isomorphic to $P_{n,n}$ whose endvertices are x and y. So $P_{n,n} \leqslant_{\mathrm{m}} \bigcup_{j=1}^n P'_j \leqslant_{\mathrm{s}} G_2 \leqslant_{\mathrm{m}} G_1 \leqslant_{\mathrm{m}} G$ (hence, $P_{n,n} \leqslant_{\mathrm{m}} G$), and the endvertices of $P_{n,n}$ are x and y. Thus, the lemma holds. $\square$

Now, we shall describe how Lemma 6.1 can be applied to 2-connected graphs and block trees. The application to 2-connected graphs, stated in Corollary 6.2 below, is more intuitive and requires less notation than the application to block trees in Corollary 6.3 that follows it.

Corollary 6.2. *Let B be a bridge of $\{x, y\}$ in a 2-connected graph G, for distinct vertices x and y in G. If each xy-path in B has length at least $n(n-1)$, and if each xy-edge-cut in B has size at least $n2^{n^3}$, then an element of $\{C_{n,n}, C^*_{n,n}\}$ is a minor of G.*

We omit the proof of Corollary 6.2 because it is very similar to the proof of Corollary 6.3, which is presented below, and to prove Corollary 6.2 would require the introduction of a large amount of notation, as in the statement of Corollary 6.3. It will be straightforward, once Corollary 6.3 is proved, that Corollary 6.3 is, in some sense, a special case of Corollary 6.2. Now, we state and prove Corollary 6.3.

Corollary 6.3. *Let $\mathscr{T} = (\mathscr{G}, T)$ be a block tree. For each link ε of T, consider the partial composition $\mathscr{T}_\varepsilon = (\{H^1_\varepsilon, H^2_\varepsilon\}, T/(E(T)-\varepsilon))$ of $\mathscr{T}$, and, for each $i \in \{1,2\}$, let $h^i_\varepsilon \,||\, u^i_\varepsilon v^i_\varepsilon$ denote the edge of H^i_ε labeled ε. If there are a link $\varepsilon \in E(T)$, an index $i \in \{1,2\}$, and an integer n exceeding 3, such that each $u^i_\varepsilon v^i_\varepsilon$-path in $H^i_\varepsilon \backslash h^i_\varepsilon$ has length at least $n(n-1)$ and each $u^i_\varepsilon v^i_\varepsilon$-edge-cut in $H^i_\varepsilon \backslash h^i_\varepsilon$ has size at least $n2^{n^3}$, then $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$ or $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$.*

Proof. Assume that the link ε of T and the integers i and n satisfy the hypotheses. Since $\mathscr{T}$ is a block tree, H^i_ε is 2-connected. By Lemma 6.1, either $C^*_{n,n} \leqslant_{\mathrm{m}} H^i_\varepsilon \backslash h^i_\varepsilon$, or $P_{n,n} \leqslant_{\mathrm{m}} H^i_\varepsilon \backslash h^i_\varepsilon$ and the endvertices of $P_{n,n}$ are u^i_ε and v^i_ε. Since H^i_ε is the composition of one of the restrictions of $\mathscr{T}$ induced by one of the components of $T \backslash \varepsilon$, it follows from Lemma 4.3 that $H^i_\varepsilon \leqslant_{\mathrm{m}} G(\mathscr{T})$. Consequently, $C^*_{n,n} \leqslant_{\mathrm{m}} G(\mathscr{T})$ or $P_{n,n} \cup h^i_\varepsilon \leqslant_{\mathrm{m}} G(\mathscr{T})$. Since $P_{n,n}$ and h^i_ε each have u^i_ε and v^i_ε as endvertices, $(P_{n,n} \cup h^i_\varepsilon)/h^i_\varepsilon \cong C_{n,n}$. The result follows. $\square$

7. *n*-Close block trees

In this section, we shall concentrate on graphs that do not satisfy the hypotheses of Corollary 6.3. We formalize this as follows. Let n be an integer exceeding three, and let $\mathscr{T} = (\mathscr{G}, T)$ be a block tree. For each link ε of T, let $\mathscr{T}_\varepsilon$ denote the partial composition $(\{H^1_\varepsilon, H^2_\varepsilon\}, T/(E(T)-\varepsilon))$ of $\mathscr{T}$, and, for each $i \in \{1,2\}$, let $h^i_\varepsilon \,||\, u^i_\varepsilon v^i_\varepsilon$ denote the edge of H^i_ε labeled ε. We call $\mathscr{T}$ an *n-close* block tree if for every link ε of T and each $i \in \{1,2\}$ at least one of the following holds:

(i) Every $u^i_\varepsilon v^i_\varepsilon$-path in $H^i_\varepsilon \backslash h^i_\varepsilon$ has length less than $n(n-1)$.
(ii) Every $u^i_\varepsilon v^i_\varepsilon$-edge-cut in $H^i_\varepsilon \backslash h^i_\varepsilon$ has size less than $n2^{n^3}$.

We have already seen in Corollary 6.3 that if the 3-block tree of a 2-connected graph has a 3-connected node graph with a cycle of length at least N, where N is the number from Theorem 3.7 that depends on n, then F_n is a minor of G. Also, we have seen in Theorem 5.1 that if the tree of the 3-block tree of G contains a path of length at least $4(n-1)+1$, then F_n is a minor of G. Additionally, we have seen in Corollary 4.4 that if the 3-block tree of G is not n-close, for some integer n exceeding 3, then $C_{n,n}$ or $C^*_{n,n}$ is a minor of G. So, we may restrict our attention to an arbitrary n-close 3-block tree $\mathscr{T}$ whose tree has no path of length exceeding $4(n-1)$ and whose 3-connected node graphs have no cycles of length exceeding N, where $n>3$ and N is the number from Theorem 3.7 depending on n. In this section, we shall show that if $\mathscr{T}$ is such a 3-block tree, then the type of $G(\mathscr{T})$ is bounded from above by a function of n, or $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$, or $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$.

Before we can state and prove any results in this section, we need to make some definitions and assumptions, and develop some terminology. By a *rooted edge-sum tree* we mean an edge-sum tree $\mathscr{T}=(\mathscr{G},T)$ whose tree T is a *rooted tree* (that is, T contains a distinguished node ξ called the *root* of T). If H is the node graph in $\mathscr{G}$ that corresponds to ξ, then call H the *root graph* of $\mathscr{T}$. The *depth* of T, denoted $D(T)$, is $\max\{d_T(\xi,\eta)\colon \eta\in V(T)\}$, where $d_T(\xi,\eta)$ is the distance in T between the root ξ of T and η. We will sometimes abuse terminology and notation by referring to the root and the depth of $\mathscr{T}$ rather than to the root and the depth of T.

It is easy to see that if T has no path of length exceeding $2M$, where M is a nonnegative integer, then, by distinguishing an appropriate vertex of T, the edge-sum tree $\mathscr{T}$ can be viewed as a rooted edge-sum tree of depth at most M.

As noted earlier, we may restrict our attention to an arbitrary n-close 3-block tree whose tree has no path of length exceeding $4(n-1)$ and whose 3-connected node graphs have no cycles of length exceeding $N_n=N$, where n is an integer exceeding 3 and N is the number form Theorem 3.7 depending on n. Clearly, if we think of such a 3-block tree as being rooted, then we may view it as having depth at most $M_n=2(n-1)$. We shall see that these values M_n and N_n, that depend only on an integer n that exceeds 3, appear in several of the results of this section.

Let n be an integer exceeding three. If $\mathscr{T}$ is an edge-sum tree with the properties (i) and (ii) below, then call $\mathscr{T}$ a $(d,c;n)$-*edge-sum tree*. Furthermore, if $\mathscr{T}$ is a block tree or a 3-block tree, then call $\mathscr{T}$ a $(d,c;n)$-*block tree* or a $(d,c;n)$-3-*block tree*, respectively.

(i) $\mathscr{T}$ can be viewed as a rooted block tree of depth at most d, for some nonnegative integer d that does not exceed M_n.
(ii) Each block of each node graph of $\mathscr{T}$ either has no cycle of length exceeding N_n or is a cycle. Moreover, if $D(\mathscr{T})=d$, and B is a block of the root graph of $\mathscr{T}$ that is not a cycle, then B has no cycle of length exceeding c, for some integer c such that $1<c\leqslant N_n$.

If $\mathscr{T}$ is a $(d,c;n)$-edge-sum tree, and each node graph different from the root graph is 2-connected, then call $\mathscr{T}$ a $(d,c;n)$-*near-block tree*. If $\mathscr{T}$ is a $(d,c;n)$-block tree or a $(d,c;n)$-3-block tree, and $\mathscr{T}$ is n-close, then call $\mathscr{T}$ a $(d,c;n)$-*close block tree* or a $(d,c;n)$-*close* 3-*block tree*, respectively. In particular, each $(0,c;n)$-block tree is a $(0,c;n)$-close block tree. Note that if $2 \leqslant c' \leqslant c \leqslant N_n$, then each $(d,c';n)$-edge-sum tree is a $(d,c;n)$-edge-sum tree. Also, note that if $0 \leqslant d' < d \leqslant M_n$, then each $(d',N_n;n)$-edge-sum tree is a $(d,c;n)$-edge-sum tree.

Now, we are ready to state the main result Theorem 7.1 of this section. The statement of Theorem 7.1 will be followed by several lemmas that will be used in its proof.

Theorem 7.1. *Let $\mathscr{T}=(\mathscr{G},T)$ be a $(d,c;n)$-close 3-block tree for some integer n exceeding three. Then one of the following holds*:

(i) $t(G(\mathscr{T})) < F(n)$, *where* $F(n)=n^3(N_n+1)^4/16+2^{n^3}n^3(N_n+1)^3/3+N_n(N_n+1)/2$.
(ii) $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$ *or* $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$.

The first lemma, which is stated without proof, describes a well-known property of edge-cuts in connected graphs. In the lemma that follows it, we shall show that, for each block B of $G(\mathscr{T})$ that contains more than one edge, there is a $(d,c;n)$-block tree $\mathscr{T}_B$, called a *block-tree reduction of B in $\mathscr{T}$*, such that $G(\mathscr{T}_B)=B$.

Lemma 7.2. *If G is a connected graph and S is an xy-edge-cut in G, then $G\backslash S$ is made up of two components, C_x containing x and C_y containing y.*

Lemma 7.3. *Let $\mathscr{T}_G=(\mathscr{G},T)$ be a $(d,c;n)$-edge-sum tree whose composition is G, where n is an integer that exceeds 3. If B is a block of G that contains at least one edge, then there is a $(d,c;n)$-edge-sum tree, namely, $\mathscr{T}_B$, whose composition is B. In particular, if B is 2-connected, then there is a $(d,c;n)$-block tree $\mathscr{T}_B$ whose composition is B. Moreover, if $e \in E(B)$ belongs to the root graph of $\mathscr{T}_G$, then e belongs to the root graph of $\mathscr{T}_B$.*

Proof. If B is a link-edge of G or a cycle of G, then it follows that $\mathscr{T}_B=(\{B\},K_1)$ is a $(0,2;n)$-edge-sum tree. It is trivial that $\mathscr{T}_B$ satisfies the remaining conditions stated in the lemma. So, for the remainder of the proof, we may assume that B is 2-connected and not a cycle.

If $D(T)=0$, then $\mathscr{T}_G=(\{G\},K_1;n)$, and $\mathscr{T}_G$ is a $(0,c;n)$-edge-sum tree. It follows that B has no cycles of length exceeding c. Then $\mathscr{T}_B=(\{B\},K_1;n)$ is a $(0,c;n)$-block tree. The remaining condition of the lemma is satisfied since $D(\mathscr{T}_B)=0$.

Now, we may assume that d is a positive integer, and that the lemma holds for all $(d-1,N_n;n)$-edge-sum trees. Consider $\mathscr{T}=\mathscr{T}_G/(E(G)-E(B))$. By Lemma 4.6, $G(\mathscr{T})=G/(E(G)-E(B))$, which is 1-isomorphic to B. Let ξ denote the root of T, and let H denote the root graph of $\mathscr{T}$. Let us consider the star $\mathscr{T}_*$ of $\mathscr{T}$ at H as defined in Section 4 and use the notation introduced in that definition.

If there is some $j \in [m]_+$ so that the set of node graphs of $\mathscr{T}_*/\{\varepsilon_i\colon i \in ([m]_+ - \{j\})\}$ is made up of K^j and a graph H_0 whose set of edges consists of a single edge, namely

h_j, then consider the edge-sum tree $\mathscr{T}'_j = \mathscr{T}/(E(T) - E(T_j))$. The tree $\mathscr{T}'_j$, which is isomorphic to T_j, is obtained by contracting $E(T) - E(T_j)$ in T to the node ξ_j; let us view $\mathscr{T}'_j$ as being rooted at ξ_j. Since $\mathscr{T}'_j$ is a partial composition of $\mathscr{T}$, it follows that $G(\mathscr{T}'_j) = G(\mathscr{T}) \cong_1 B$. The set of node graphs of $\mathscr{T}'_j$ is obtained from the set of node graphs of $\mathscr{T}_j$ by replacing the root graph H_j of $\mathscr{T}_j$ by H_j/k_j if h_j is a loop, and by $H_j \backslash k_j$ if h_j is not a loop. It follows that $\mathscr{T}'_j$ is a rooted edge-sum tree of depth less than d whose composition is 1-isomorphic to B, and that each block of each node graph of $\mathscr{T}'_j$ either lacks a cycle of length exceeding N_n or is a cycle. Hence, $\mathscr{T}'_j$ is a $(d-1, N_n; n)$-edge-sum tree, and, by hypothesis, there is a $(d, c; n)$-block tree $\mathscr{T}_B$ of B, and the result holds.

Finally, we may assume that the node graph H'_j of $\mathscr{T}_*/\{\varepsilon_i: i \in ([m]_+ - \{j\})\}$ that is not K^j has at least two edges (hence, at least one unlabeled edge), for each $j \in [m]_+$. It follows that all edges of K^i belong to a single block of K^i, for each $i \in [m]_+$; otherwise, $G(\mathscr{T})$ would have more than one block containing edges. For each $i \in [m]_+$, if K^i consists of a single edge, then contract the link ε_i in $\mathscr{T}_*$. Let $\mathscr{T}' = (\mathscr{G}', T')$ denote the resulting rooted edge-sum tree, and let H' denote the root graph of $\mathscr{T}'$. It follows that each node graph K' in $\mathscr{G}' - H'$ is a 2-connected graph with, perhaps, some isolated vertices. It then follows that H' is a 2-connected graph with, perhaps, some isolated vertices; otherwise, $G(\mathscr{T})$ would have more than one block containing edges. Note that, for each node graph K' in $\mathscr{G}' - H'$, the graph K' is the composition of the edge-sum tree $\mathscr{T}_i$ rooted at ξ_i, for some $i \in [m]_+$, and the edge $k_i \in E(K')$ belongs to the root graph of $\mathscr{T}_i$. By hypothesis, for each node graph K' in $\mathscr{G}' - H'$, there is a $(d-1, N_n; n)$-block tree $\mathscr{T}_{K'}$ whose composition is K' and whose root graph contains k_i. Consider the rooted edge-sum tree $\mathscr{T}^*$ defined as follows. Let $H^* = H'[E(H')]$ be the root graph of $\mathscr{T}^*$, let the directed labeling of H^* in $\mathscr{T}^*$ agree with the directed labeling of H' in $\mathscr{T}'$, and let ξ^* denote the root of the tree T^* of $\mathscr{T}^*$. We obtain T^* by connecting ξ^* to the root of the tree of $\mathscr{T}_{K'}$ with a link, for each K' in $\mathscr{G}' - H'$. If h_i is a labeled edge of H^*, then there is a $(d-1, N_n; n)$-block tree $\mathscr{T}_{K'}$, for some K' in $\mathscr{G}' - H'$, whose root graph contains k_i. Assign the label ε_i to k_i, and direct k_i in $\mathscr{T}^*$ so that its direction agrees with its direction in $\mathscr{T}'$. Note that H^* is obtained by deleting all isolated vertices from H'. Also, note that H' is obtained by edge-summing H with graphs K^i each consisting of a single edge, which amounts to deleting or contracting an edge of H, depending on whether such a graph K^i is a link-edge or a loop. Lastly, note that H is obtained by contracting a set of edges in the root graph H_G of $\mathscr{T}_G$, and thus $H^* \leqslant_{\mathrm{m}} H_G$. Since each block of H_G is a cycle or contains no cycle of length exceeding c, and since H' is a 2-connected graph with, perhaps, some isolated vertices, it follows that H^* is a cycle, or H^* is a block that contains no cycle of length exceeding c. It follows that $\mathscr{T}^*$ is a $(d, c; n)$-block tree whose composition is B, as required. □

We shall prove Theorem 7.1 by induction on the indices d and c. The next few lemmas will handle the details of certain steps of the induction in order to make the proof of Theorem 7.1 shorter and more readable.

Lemma 7.4. *Let $\mathscr{T}=(\mathscr{G},L_{\mathscr{G}},T)$ be a $(0,c;n)$-close block tree, for some integer n exceeding* 3. *Then $t(G(\mathscr{T}))\leqslant c(c+1)/2$.*

Proof. Since $D(T)=0$, it follows that $\mathscr{G}$ contains only one node graph H, which is an unlabeled 2-connected graph, and $G(\mathscr{T})=H$. Recall that $2\leqslant c\leqslant N_n$. If H is a cycle with at least 2 edges, then $t(H)=2<c(c+1)/2$. So, we may assume that H is a 2-connected graph, each cycle of which has length at most c. By Corollary 3.6, $t(H)\leqslant c(c+1)/2$. $\square$

Lemma 7.5. *Let $\mathscr{T}=(\mathscr{G},L_{\mathscr{G}},T)$ be a $(d,c;n)$-close block tree whose root graph is a cycle of length at least n, for some integers n and d exceeding* 3 *and* 0, *respectively. Then one of the following holds*:

(i) *There is a set $S_\backslash$ of at most $n-3$ edges in $G(\mathscr{T})$, so that if B is a 2-connected block of $G(\mathscr{T})\backslash S_\backslash$ for which $t(B)=t(G(\mathscr{T})\backslash S_\backslash)$, then there is a $(d-1,N_n;\ n)$-block tree $\mathscr{T}_B$ whose composition is B.*
(ii) $t(G(\mathscr{T}))\leqslant n-2$.
(iii) $C_{n,n-2}\leqslant_{\mathrm{m}} G(\mathscr{T})$.

Proof. Let H denote the root graph of $\mathscr{T}$, and let ξ denote the root of T. The cycle H has length N, for some integer $N\geqslant n$. We may assume that $V(H)=\{v_i\colon i\in[N]_+\}$ and that $E(H)=\{v_1v_2,v_2v_3,\ldots,v_{N-1}v_N,v_Nv_1\}$. Also, it will be convenient to think of v_1 as sometimes having the name v_{N+1}. Let us consider the star $\mathscr{T}_*$ of $\mathscr{T}$ at H as defined in Section 4 and use the notation introduced in that definition.

Since $\mathscr{T}$ is a block tree, it follows that $\mathscr{T}_i$ and $\mathscr{T}_*$ are block trees and K^i is 2-connected, for each $i\in[m]_+$. It also follows that the labeled edge k_i in K^i is a link-edge, and we shall let x_i and y_i denote the endvertices of k_i, for each $i\in[m]_+$. It follows that there are N distinct vertices in $G(\mathscr{T}_*)=G(\mathscr{T})$ corresponding to the N vertices of $V(H)$. For each $i\in[N]_+$, let the vertex in $G(\mathscr{T}_*)$ corresponding to v_i also be called v_i. Since the edges h_i and k_i are identified (and then deleted) when ε_i is contracted in $\mathscr{T}_*$, the composition $G(\mathscr{T}_*)$ is obtained from H by replacing h_i with $K^i\backslash k_i$ so that x_i is identified with one endvertex of h_i and y_i is identified with the other endvertex of h_i (as determined by the directed labeling of $\mathscr{T}_*$), for each $i\in[m]_+$. Let K^i_k denote the subgraph of $G(\mathscr{T}_*)=G(\mathscr{T})$ that is isomorphic to $K^i\backslash k_i$ and replaces h_i in H, for each $i\in[m]_+$. Note that, for each $i\in[m]_+$, the graphs K^i_k and $K^i\backslash k_i$ are identical except that x_i and y_i in $K^i\backslash k_i$ are renamed in K^i_k with the endvertices of h_i in H. Fig. 9 illustrates a typical $\mathscr{T}_*$ and its composition. In this figure, the cycle whose vertex set is $\{v_i\colon i\in[6]_+\}$ is the root graph H of $\mathscr{T}_*$, and, for each $i\in[3]_+$, the node graph of $\mathscr{T}_*$ containing k_i is K^i.

First, let us assume that some edge e of H is not labeled by the directed labeling $L_{\mathscr{G}}$ of $\mathscr{T}$, and thus $e\in E(G(\mathscr{T}))$. By shifting the indices of the vertices of H, we may assume that $e=v_1v_N$. If e is deleted from $G(\mathscr{T})$, then, for each integer i such that $1<i<N$, the vertex v_i is a cut-vertex of $G(\mathscr{T})\backslash e$. It follows that each unlabeled edge

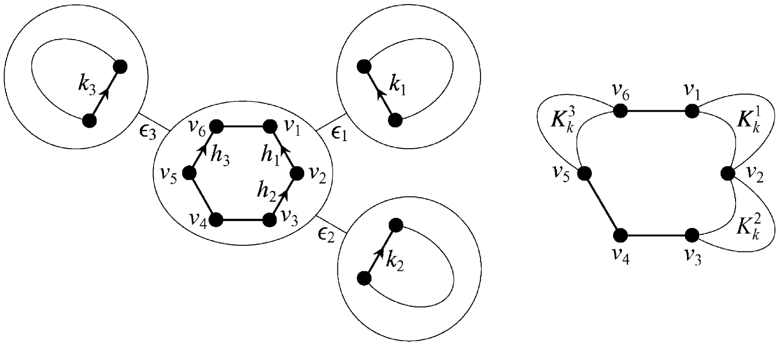

Fig. 9. A typical $\mathscr{T}_*$ and its composition $G(\mathscr{T}_*)$.

in $E(H)-e$, viewed as a subgraph of $G(\mathscr{T})\backslash e$, is a block of $G(\mathscr{T})\backslash e$, and K_k^i is a union of blocks of $G(\mathscr{T})\backslash e$, for each $i\in[m]_+$. Since K_k^i has at least one edge, for each $i\in[m]_+$, it follows that $t(G(\mathscr{T})\backslash e)=\max\{t(K_k^i)\colon i\in[m]_+\}$. Let $l\in[m]_+$ be an index for which $t(G(\mathscr{T})\backslash e)=t(K_k^l)$, and let B be a block of K_k^l for which $t(K_k^l)=t(B)$. If $|E(B)|=1$, then it follows that each block of $G(\mathscr{T})\backslash e$ is a single edge, and hence $t(G(\mathscr{T}))\leqslant|\{e\}|+t(B)=2\leqslant n-2$. So, we may assume that B has more than one edge, and hence B is 2-connected. Note that $K_k^l\cong K^l\backslash k_l=G(\mathscr{T}_l\backslash k_l)$. Since $\mathscr{T}_l\backslash k_l$ is a $(d-1,N_n;n)$-edge-sum tree, by Lemma 7.3, there is a $(d-1,N_n;n)$-block tree whose composition is B, as required.

For the rest of the proof, we may assume that each edge of H is labeled by $L_{\mathscr{G}}$, and consequently $m=N$. By an appropriate permutation of $[N]_+$ applied to the index i in ε_i, h_i, k_i, $\mathscr{T}_i$, T_i, K^i, and K_k^i, we may assume that $h_i=v_iv_{i+1}$ in H.

For the next case, which is similar to the first, let us assume that there is an index $l\in[N]_+$ for which K^l has an x_ly_l-edge-cut S_l^0 containing at most $n-2$ edges. By shifting the indices, we may assume that $l=N$. Clearly, $k_N\in S_N^0$. Let $S_N=S_N^0-k_N$. Then S_N is made up of unlabeled edges, and $|S_N|\leqslant n-3$. Note that $K_k^N\backslash S_N\cong K^N\backslash S_N^0$. Since K^N is 2-connected, it follows from Lemma 7.2 that $K^N\backslash S_N^0$ is made up of two components, C_x containing x_N and C_y containing y_N. Thus, $K_k^N\backslash S_N$ is made up of two components, C_1 containing v_1 and C_N containing v_N, and $\{C_1,C_N\}$ are $\{C_x,C_y\}$ identical except for the names of v_1 and v_N in $\{C_x,C_y\}$. It follows that $G(\mathscr{T})\backslash S_N=G(\mathscr{T}_*)\backslash S_N$ is as in Fig. 10.

Note that, for each integer i such that $1<i<N$, the vertex v_i is a cut-vertex of $G(\mathscr{T})\backslash S_N$ (see Fig. 10). Furthermore, if C_1 is not isomorphic to K_1, then C_1 is a union of bridges of v_1 in $G(\mathscr{T})\backslash S_N$. Similarly, if C_N is not isomorphic to K_1, then C_N is a union of bridges of v_N in $G(\mathscr{T})\backslash S_N$. It follows that K_k^i is a union of blocks of $G(\mathscr{T})\backslash S_N$ for each $i\in[N-1]_+$, and, C is a union of blocks of $G(\mathscr{T})\backslash S_N$, for each element C of $\{C_1,C_N\}$ that is not isomorphic to K_1. If there is an index $q\in[N-1]_+$ such that $t(K_k^q)=t(G(\mathscr{T})\backslash S_N)$, then let B be a block of K_k^q such that $t(B)=t(K_k^q)$. If B consists of a single edge, then each block of $G(\mathscr{T})\backslash S_N$ is a single edge, and hence, $t(G(\mathscr{T}))\leqslant|S_N|+t(B)\leqslant n-2$. Consequently, as before, we may

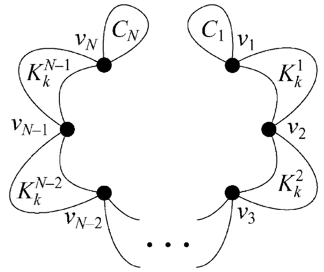

Fig. 10. A typical $G(\mathscr{T})\backslash S_N$.

assume that B is 2-connected. Recall that $K_k^q \cong K^q\backslash k_q = G(\mathscr{T}_q\backslash k_q)$. Since $\mathscr{T}_q\backslash k_q$ is a $(d-1, N_n; n)$-edge-sum tree, by Lemma 7.3, there is a $(d-1, N_n; n)$-block tree whose composition is B, as required. If there is no index $q \in [N-1]_+$ for which $t(K_k^q) = t(G(\mathscr{T})\backslash S_N)$, then $t(C_1 \dot\cup C_N) = t(G(\mathscr{T})\backslash S_N)$. Let B be a block of $C_1 \dot\cup C_N$ for which $t(B) = t(C_1 \dot\cup C_N)$. As before, $t(G(\mathscr{T})) \leqslant |S_N| + t(B) \leqslant n-2$ if B consists of an edge; so we may assume that B is 2-connected. Note that $C_1 \dot\cup C_N \cong K^N\backslash S_N^0 = G(\mathscr{T}_N\backslash S_N^0)$. Since $\mathscr{T}_N\backslash S_N^0$ is a $(d-1, N_n; n)$-edge-sum tree, by Lemma 7.3, there is a $(d-1, N_n; n)$-block tree whose composition is B, as required.

For the final case, let us assume, for each $i \in [N]_+$, that every $x_i y_i$-edge-cut in K^i has at least $n-1$ edges. The following holds for each $i \in [N]_+$. Let S_i^0 be an $x_i y_i$-edge-cut in K^i. Clearly, $k_i \in S_i^0$, and the edges in $S_i^0 - k_i$ are unlabeled in K^i. Let $S_i = S_i^0 - k_i$. Then $|S_i| \geqslant n-2$. By Lemma 7.2, $K^i\backslash S_i^0 \cong K_k^i\backslash S_i$ consists of two components, $C_{i,1}$ containing v_i and $C_{i,2}$ containing v_{i+1}. It is straightforward that S_i is a $v_i v_{i+1}$-edge-cut of K_k^i. From this it follows that $C_{i,1} \cup C_{i,2} \cup s$ is connected, for each $s \in S_i$; in particular, each $s \in S_i$ has one endvertex in $V(C_{i,1})$ and the other endvertex in $V(C_{i,2})$. Next, we show that a multi-edge of size at least $n-2$ is a minor of K_k^i.

Consider $K_k^i/E(C_{i,1} \dot\cup C_{i,2})$, for any $i \in [N]_+$. We can see that this graph is isomorphic to a multi-edge of size at least $n-2$ with endvertices v_i and v_{i+1} as follows. When $E(C_{i,1})$ is contracted in K_k^i, we may identify all of the vertices of $C_{i,1}$ to v_i. Similarly, when $E(C_{i,2})$ is contracted in K_k^i, we may identify all of the vertices of $C_{i,2}$ to v_{i+1}. Since $C_{i,1}$ and $C_{i,2}$ are disjoint, v_i and v_{i+1} are distinct vertices in $K_k^i/E(C_{i,1} \dot\cup C_{i,2})$. Hence, in $K_k^i/E(C_{i,1} \dot\cup C_{i,2})$, one endvertex of s is v_i and the other endvertex of s is v_{i+1}, for each $s \in S_i$. Consequently, $K_k^i/E(C_{i,1} \dot\cup C_{i,2})$ is a multi-edge of size at least $n-2$ with endvertices v_i and v_{i+1}, for each $i \in [N]_+$.

Since a multi-edge of size at least $n-2$ with endvertices v_i and v_{i+1} is a minor of K_k^i, for each $i \in [N]_+$, it follows that $C_{N,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$. Since $N \geqslant n$, it follows that $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$, as required. $\square$

In Lemma 4.6 we saw that, given an edge-sum tree $\mathscr{T}$ and disjoint sets C and D of edges in $G(\mathscr{T})$, contracting each edge of C in its appropriate node graph and deleting each edge of D in its appropriate node graph is equivalent to first taking the composition of $\mathscr{T}$ and then performing contractions on the edges of C and deletions on the edges of D. In order to prove the next lemma, we would like, in some sense, to be able to perform a contraction or a deletion on a labeled edge from the root node of a near-block tree $\mathscr{T}$ and describe the effect of this on $G(\mathscr{T})$. This is described more precisely below.

Let $\mathscr{T}$ be a near-block tree of depth at least 1, and let h be a labeled edge in the root graph H of $\mathscr{T}$. Let ε denote the link of the tree T of $\mathscr{T}$ with which h is labeled, and let k denote the other edge that is labeled ε. Let $\mathscr{T}_K$ denote the restriction of $\mathscr{T}$ induced by the component T_K of $T\backslash\varepsilon$ that does not contain the root ξ of $\mathscr{T}$, and let $K = G(\mathscr{T}_K)$. It follows that $k \in E(K)$, and, since $\mathscr{T}_K$ is a block tree, K is 2-connected. Hence, k is a link-edge, and thus has distinct endvertices x_k and y_k. Since K is 2-connected, it has a cycle C_k containing k and an $x_k y_k$-edge-cut D_k containing k. Let $C_0 = E(C_k) - k$ and $D_0 = D_k - k$. Consider the partial composition $\mathscr{T}/E(T_K)$ of $\mathscr{T}$. The node graph of $\mathscr{T}/E(T_K)$ that corresponds to the endnode of ε that is not ξ is K, and K (viewed as a node graph of $\mathscr{T}/E(T_K)$) has exactly one labeled edge k. By symmetry, we may assume that the x_k and y_k are the tail and head, respectively, of k.

First, let us consider $\mathscr{T}_1 = (\mathscr{T}/E(T_K))/C_0$. Since the edges of C_0 form an $x_k y_k$-path in K, the labeled edge k is a loop in the node graph K/C_0 of $\mathscr{T}_1$. Now consider $\mathscr{T}_1/\varepsilon$. It is straightforward that $\mathscr{T}_1$ and $\mathscr{T}_1/\varepsilon$ are the same, except that the link ε is contracted to ξ in $\mathscr{T}_1/\varepsilon$, and the node graph in $\mathscr{T}_1/\varepsilon$ corresponding to ξ is $H_1 = H \oplus_2 (K/C_0)$. Since k is a loop in K/C_0, it follows from the definition of edge-summing that H_1 is 1-isomorphic to the disjoint union of H/h and K/C_k. Also note that K/C_k, viewed as a subgraph of H_1, has no labeled edges and is a union of blocks of H_1 (provided that K/C_k has at least one edge). It is straightforward that $G(\mathscr{T}_1/\varepsilon)$ is 1-isomorphic to the disjoint union of K/C_k and $G(\mathscr{T}_{/h})$, where $\mathscr{T}_{/h}$ is obtained by contracting h in the root graph H of the restriction $\mathscr{T} - V(T_K)$ of $\mathscr{T}$. Note that K/C_k is the composition of $\mathscr{T}_K/C_k$. Let us abbreviate $\mathscr{T}_K/C_k$ as $\mathscr{T}_{/C_k}$. It follows from the way that $\mathscr{T}_1$, $\mathscr{T}_{/h}$, and $\mathscr{T}_{/C_k}$ are defined that $G(\mathscr{T}/C_0)$ is 1-isomorphic to the disjoint union of the compositions of $\mathscr{T}_{/h}$ and $\mathscr{T}_{/C_k}$. Thus, $t(G(\mathscr{T})/C_0) = t(G(\mathscr{T}/C_0)) = \max\{t(G(\mathscr{T}_{/h})), t(G(\mathscr{T}_{/C_k}))\}$, and hence $t(G(\mathscr{T})) \leqslant |C_0| + \max\{t(G(\mathscr{T}_{/h})), t(G(\mathscr{T}_{/C_k}))\}$. Note that $\mathscr{T}_{/C_k}$ is an edge-sum tree of depth less than $D(\mathscr{T})$ and that $\mathscr{T}_{/h}$ is a near-block tree. So let us say that we can *essentially contract* a labeled edge h in the root graph H of a near-block tree $\mathscr{T}$ by contracting C_0 in $\mathscr{T}$, and, after essentially contracting h, it is sufficient to consider $\{\mathscr{T}_{/h}, \mathscr{T}_{/C_k}\}$, as described above. The process of essentially contracting the labeled edge h in H in $\mathscr{T}$ is illustrated in Fig. 11.

Now, consider $\mathscr{T}_2 = (\mathscr{T}/E(T_K))\backslash D_0$. Since D_k is an $x_k y_k$-edge-cut in K, it follows from Lemma 7.2 that k is a cut-edge of $K\backslash D_0$ whose deletion results in components K_{x_k} and K_{y_k} containing x_k and y_k, respectively (hence, $K\backslash D_k = K_{x_k} \dot\cup K_{y_k}$). It follows that $\mathscr{T}_2$ and $\mathscr{T}_2/\varepsilon$ are the same, except that ε is contracted to ξ in $\mathscr{T}_2/\varepsilon$, and the node graph in $\mathscr{T}_2/\varepsilon$ corresponding to ξ is $H_2 = H \oplus_2 (K\backslash D_0)$. Note that, whether h is a loop or a link-edge, $H_2 \cong_1 (H\backslash h)\dot\cup(K\backslash D_k)$. Also note that each component K' of $K\backslash D_k$, viewed as a subgraph of H_2, is an unlabeled union of blocks of H_2, provided that $E(K') \neq \emptyset$. It follows that $G(\mathscr{T}_2/\varepsilon) \cong_1 (K\backslash D_k)\dot\cup G(\mathscr{T}_{\backslash h})$, where $\mathscr{T}_{\backslash h}$ is obtained by deleting h from $\mathscr{T} - V(T_K)$. Note that $K\backslash D_k = G(\mathscr{T}_K\backslash D_k)$, and

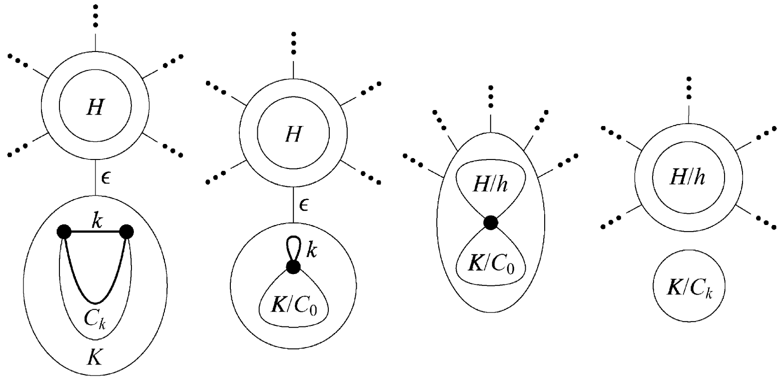

Fig. 11. $\mathscr{T}_{/h}$ and $G(\mathscr{T}_{/C_k}) = K/C_k$ are obtained by essentially contracting h.

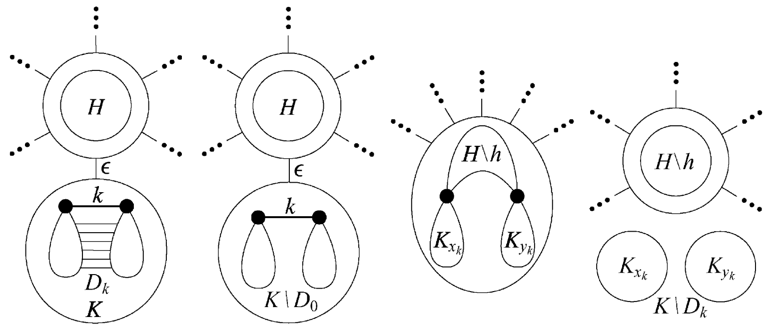

Fig. 12. $\mathscr{T}_{\backslash h}$ and $G(\mathscr{T}_{\backslash D_k}) = K\backslash D_k$ are obtained by essentially deleting h.

abbreviate $\mathscr{T}_K\backslash D_k$ as $\mathscr{T}_{\backslash D_k}$. It follows from the way that $\mathscr{T}_2$, $\mathscr{T}_{\backslash h}$, and $\mathscr{T}_{\backslash D_k}$ are defined that $G(\mathscr{T}\backslash D_0) \cong_1 G(\mathscr{T}_{\backslash h})\dot{\cup}G(\mathscr{T}_{\backslash D_k})$. Thus, $t(G(\mathscr{T})) \leqslant |D_0| + t(G(\mathscr{T})\backslash D_0) = |D_0| + t(G(\mathscr{T}\backslash D_0)) = |D_0| + \max\{t(G(\mathscr{T}_{\backslash h})), t(G(\mathscr{T}_{\backslash D_k}))\}$. Note that $\mathscr{T}_{\backslash D_k}$ is an edge-sum tree of depth less than $D(\mathscr{T})$ and that $\mathscr{T}_{\backslash h}$ is a near-block tree. So we can *essentially delete* a labeled edge h from the root graph H of a near-block tree $\mathscr{T}$ by deleting D_0 from $\mathscr{T}$, and, after essentially deleting h, we may consider $\{\mathscr{T}_{\backslash h}, \mathscr{T}_{\backslash D_k}\}$, as described above. The process of essentially deleting a labeled link-edge h from H in $\mathscr{T}$ is illustrated in Fig. 12.

Finally, we extend the definition to disjoint sets C and D of labeled edges in the root graph H of a near-block tree $\mathscr{T}$ so that we essentially contract C and essentially delete D. Consider the star $\mathscr{T}_*$ of $\mathscr{T}$ at H. Recall that $\{h_i: i \in [m]_+\}$ is the set of labeled edges in the root graph H of $\mathscr{T}$. Since $\mathscr{T}$ is a near-block tree, it follows that $\mathscr{T}_i$ is a block tree and that K^i is 2-connected, for each $i \in [m]_+$. Hence, K^i contains a cycle C_{k_i} containing k_i and an $x_{k_i}y_{k_i}$-edge-cut D_{k_i}, where x_{k_i} and y_{k_i} are the tail and head, respectively, of k_i as determined by the directed labeling of $\mathscr{T}_*$, for each $i \in [m]_+$. Note that there are subsets I_C and I_D of $[m]_+$ so that $C = \{h_i: i \in I_C\}$ and $D = \{h_i: i \in I_D\}$.

Let $C_i = C_{k_i} - k_i$ for each $i \in I_C$, and let $D_i = D_{k_i} - k_i$ for each $i \in I_D$. Let $\mathscr{T}_{/C_{k_i}} = \mathscr{T}_i / C_{k_i}$ for each $i \in I_C$, let $\mathscr{T}_{\backslash D_{k_i}} = \mathscr{T}_i \backslash D_{k_i}$ for each $i \in I_D$, and let $\mathscr{T}_{/C\backslash D}$ be the near-block tree that is obtained from the restriction $\mathscr{T} - \bigcup_{i \in (I_C \cup I_D)} V(T_i)$ of $\mathscr{T}$ by replacing its root graph H with $H/C\backslash D$. Let us consider the collection $\mathfrak{T} = \{\mathscr{T}_{/C\backslash D}\} \cup \{\mathscr{T}_{/C_{k_i}}: i \in I_C\} \cup \{\mathscr{T}_{\backslash D_{k_i}}: i \in I_D\}$ of edge-sum trees. It is straightforward that the disjoint union of the compositions of the elements of $\mathfrak{T}$ is 1-isomorphic to $G(\mathscr{T}) / \bigcup_{i \in I_C} C_i \backslash \bigcup_{i \in I_D} D_i$. It follows that $t(G(\mathscr{T})) \leqslant |\bigcup_{i \in I_C} C_i| + |\bigcup_{i \in I_D} D_i| + \max\{t(G(\mathscr{U})): \mathscr{U} \in \mathfrak{T}\}$. So let us say that we can essentially contract C in and essentially delete D from the root graph H of a near-block tree $\mathscr{T}$ by contracting $\bigcup_{i \in I_C} C_i$ in and by deleting $\bigcup_{i \in i_D} D_i$ from $\mathscr{T}$. After essentially contracting C and essentially deleting D, it is sufficient to consider $\mathfrak{T}$, as described above.

Lemma 7.6. *Let $\mathscr{T} = (\mathscr{G}, T)$ be a $(d, c; n)$-close block tree whose root graph H contains no cycle of length exceeding c, for some integers n, c, and d exceeding, respectively, 3, 1, and 0, and assume that each edge of H is labeled. Then one of the following holds*:

(i) *There are disjoint subsets $E_/$ and $E_\backslash$ of $E(G(\mathscr{T}))$ containing fewer than $c^3n^2/8$ edges and $2^{n^3-1}c^2n^2$ edges, respectively, so that if B is a 2-connected block of $G(\mathscr{T})/E_/\backslash E_\backslash$ for which $t(B) = t(G(\mathscr{T})/E_/\backslash E_\backslash)$, then there is a $(d, c-1; n)$-block tree if $c \geqslant 3$ or a $(d-1, N_n; n)$-block tree if $c = 2$, whose composition is B.*
(ii) $t(G(\mathscr{T})) < c^3n^2/8 + 2^{n^3-1}c^2n^2 + 1$.
(iii) $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$.

Proof. Let $\mathscr{T}_*$ be the star of $\mathscr{T}$ at H. For each $i \in [m]_+$, we assign a weight of s or l to $h_i \in E(H)$ as follows. If every cycle in K^i that contains k_i has length exceeding $n(n-1)$, then let the weight $w(h_i)$ of h_i be l; otherwise, let $w(h_i) = s$.

Let C be a longest cycle of H. Clearly, $|E(C)| = c$. For each pair $\{u, v\}$ of vertices of C, let P_{uv} be a uv-path in H made up of edges weighted s such that $V(P_{uv}) \cap V(C) = \{u, v\}$, if such a path exists; otherwise, let P_{uv} be the subgraph of H made up of the vertices u and v. Let $P_s = \bigcup_{\{u,v\} \subseteq V(C)} P_{uv}$, and let F_s be a spanning forest of P_s, and hence $|E(F_s)| \leqslant |E(P_s)|$. Note that if P_{uv} is a path, then the length of P_{uv} is at most the distance between u and v in C, since C is a longest cycle in H. It follows that

$$|E(F_s)| \leqslant \begin{cases} c\displaystyle\sum_{i=1}^{(c-1)/2} i = c\frac{((c-1)/2)(c+1)/2}{2} = \frac{c(c^2-1)}{8} < \frac{c^3}{8} & \text{if } n \text{ is odd;} \\ c\displaystyle\sum_{i=1}^{c/2-1} i + \frac{c}{2}\frac{c}{2} = c\frac{c/2(c/2-1)}{2} + \frac{c^2}{4} = \frac{c^3}{8} & \text{if } n \text{ is even.} \end{cases}$$

Let I_F be the set of indices in $[m]_+$ for which $h_i \in F_s$. For each $i \in I_F$, let C_{k_i} be a cycle in K^i containing k_i whose length is at most $n(n-1)$, and let $C_i = E(C_{k_i}) - k_i$. Then C_i consists of unlabeled edges for each $i \in I_F$. Let us essentially contract F_s in

H by contracting $C_s = \bigcup_{i\in I_F} C_i$ in $\mathscr{T}$. Note that $|C_s| \leqslant c^3/8n(n-1) < c^3n^2/8$. So, after contracting fewer than $c^3n^2/8$ edges in $\mathscr{T}$, we may consider the collection $\mathfrak{T} = \{\mathscr{T}_{/F_s}\} \cup \{\mathscr{T}_{/C_{k_i}}: i \in I_F\}$ of edge-sum trees. Given any nonempty collection $\mathfrak{U}$ of edge-sum trees, let $G(\mathfrak{U})$ denote the disjoint union of the compositions of the elements of $\mathfrak{U}$. It follows that $G(\mathscr{T}/C_s) = G(\mathscr{T})/C_s \cong_1 G(\mathfrak{T})$. Note that $\mathscr{T}_{/F_s}$ is a $(d,c;n)$-near-block tree, and $\mathscr{T}_{/C_{k_i}}$ is a $(d-1,N_n;n)$-edge-sum tree, for each $i \in I_F$.

Let V' denote the set of vertices in the root graph $H' = H/F_s$ of $\mathscr{T}' = \mathscr{T}_{/F_s}$ corresponding to $V(C)$ in H. Clearly $|V'| \leqslant |V(C)| = c$. We consider the cases when $|V'| = 1$ and when $1 < |V'| \leqslant c$ separately.

First, assume that $|V'| = 1$. Then, the length of each cycle in H' is at most $c-1$. We can see this as follows. If C' is a longest cycle of H different from C, then, by Corollary 3.5, C and C' have at least two vertices in common. When F_s is contracted in H, the vertices v_1 and v_2 are identified to a single vertex, and thus the subgraph of H' corresponding to C' is an edge-disjoint union of cycles of length less than c. It follows that $\mathscr{T}'$ is a $(d,c-1;n)$-near-block tree if $c > 2$. If $c = 2$, then, since H is a 2-connected graph, each block of H' is a loop. It follows that $G(\mathscr{T}')$ is 1-isomorphic to $\bigcup_{i\in I_{H'}} G(\mathscr{T}_i/k_i)$, where $I_{H'} = \{i: h_i \in H'\}$. If $c > 2$, then let $\mathfrak{T}_0 = \mathfrak{T}$; if $c = 2$, then let $\mathfrak{T}_0 = (\mathfrak{T} - \{\mathscr{T}'\}) \cup \{\mathscr{T}_i/k_i: i \in I_{H'}\}$. Note that $\mathscr{T}_i/k_i$ is a $(d-1,N_n;n)$-edge-sum tree, for each $i \in I_{H'}$. It follows that $\mathfrak{T}_0$ is made up of a number of $(d-1,N_n;n)$-edge-sum trees and, if $c > 2$, it consists of a single $(d,c-1;n)$-near-block tree. It is straightforward that $G(\mathscr{T}/C_s) \cong_1 G(\mathfrak{T}_0)$.

If $t(G(\mathscr{T}/C_s)) = 1$, then $t(G(\mathscr{T})) \leqslant |C_s| + t(G(\mathscr{T}/C_s)) < c^3n^2/8 + 1$, in which case the result follows. If $t(G(\mathscr{T}/C_s)) = t(G(\mathscr{T})/C_s) > 1$, then there is a 2-connected block B for which $t(B) = t(G(\mathscr{T}/C_s))$. Since $G(\mathscr{T}/C_s) \cong_1 G(\mathfrak{T}_0)$, the block B is isomorphic to a block of $G(\mathscr{T}_0)$, for some $\mathscr{T}_0 \in \mathfrak{T}_0$. By Lemma 7.3, there is a $(d,c-1;n)$-block tree or, if $c = 2$, a $(d-1,N_n;n)$-block tree whose composition is B. So, if we let $E_/ = C_s$ and $E_\backslash = \emptyset$, the lemma follows.

Now, let us consider the case in which $1 < |V'| \leqslant c$. Note that H' is connected since H is connected. So, for each pair $\{u,v\}$ of vertices in V', there is a uv-path in H', but we can see that no uv-path in H' consists only of edges weighted s, as follows. If there were a uv-path in H' consisting only of edges weighted s, then H would contain, for some $\{u_0,v_0\} \subseteq V(C)$, a u_0v_0-path consisting only of edges weighted s. It follows that F_s would contain a u_0v_0-path; consequently, u_0 and v_0 would be identified to the same vertex when contracting F_s in H; a contradiction. So, for each pair $\{u,v\}$ of vertices in V', each uv-path in H' contains an edge weighted l. Then H' contains a uv-edge-cut consisting only of edges weighted l, for each pair $\{u,v\}$ of vertices in V'.

If $|S_{u'v'}| \geqslant n$, for some pair $\{u',v'\}$ of vertices of V', then consider the restriction $\mathscr{T}'_*$ of the star $(\mathscr{T}')_{\mathrm{st}}$ of $\mathscr{T}'$ induced by $\{\varepsilon_i: i \in I_{u'v'}\}$, where $i \in I_{u'v'}$ if and only if $h_i \in S_{u'v'}$. By Lemma 4.3, $G(\mathscr{T}'_*) \leqslant_{\mathrm{m}} G((\mathscr{T}')_*) = G(\mathscr{T}')$. Since the weight of h_i is l, each cycle in K^i using k_i has length exceeding $n(n-1)$, for each $i \in I_{u'v'}$. Let C_{k_i} be a cycle in K^i using k_i, for each $i \in I_{u'v'}$. Consider $\mathscr{T}''_* = \mathscr{T}'_* \setminus \bigcup_{i\in I_{u'v'}} (E(K^i) - E(C_{k_i}))$. It follows that $G(\mathscr{T}''_*)$ is obtained from H' by subdividing the edge h_i with $|E(C_{k_i})| - 2$ new vertices (that is, at least $n(n-1) - 1$ new vertices), for each $i \in I_{u'v'}$, and adding, perhaps, some isolated vertices. Let P_i denote the path obtained by subdividing h_i, for each $i \in I_{u'v'}$, as just described. It follows that $H' \backslash S_{u'v'}$ and $G(\mathscr{T}''_*) \backslash \bigcup_{i \in I_{u'v'}} P_i$ are identical, except, perhaps, for additional isolated vertices in $G(\mathscr{T}''_*) \backslash \bigcup_{i \in I_{u'v'}} P_i$. By Lemma 7.2, $H' \backslash S_{u'v'}$ consists of two components $C_{u'}$ containing u' and $C_{v'}$ containing v'. Since $S_{u'v'}$ is a $u'v'$-edge-cut in H', exactly one endvertex u_i of h_i lies in $C_{u'}$, and the other endvertex v_i of h_i lies in $C_{v'}$, for each $i \in I_{u'v'}$, and hence, in $G(\mathscr{T}''_*) \backslash \bigcup_{i \in I_{u'v'}} P_i$, for each $i \in I_{u'v'}$, one endvertex of P_i lies in $C_{u'}$, and the other endvertex of P_i lies in $C_{v'}$. Let us contract $C_{u'}$ to a single vertex u^* and $C_{v'}$ to a single vertex v^* in $G(\mathscr{T}''_*)$. Since P_i is obtained by subdividing h_i in H', for each $i \in I_{u'v'}$, it follows that $G_0 = G(\mathscr{T}''_*)/E(C_{u'} \cup C_{v'})$ is made up of $|S_{u'v'}| \geqslant n$ pairwise internally vertex-disjoint u^*v^*-paths, each having length at least $n(n-1)$, and, perhaps, some isolated vertices. Hence, $C^*_{n,n-2} \leqslant_{\mathrm{m}} G_0 \leqslant_{\mathrm{m}} G(\mathscr{T}''_*) \leqslant_{\bar{s}} G(\mathscr{T}'_*) \leqslant_{\mathrm{m}} G(\mathscr{T}') \leqslant_{\mathrm{m}} G(\mathscr{T})$, and the lemma holds.

It remains to consider the case when $|S_{uv}| < n$, for each pair $\{u,v\} \subseteq V'$. Let $S_l = \bigcup_{\{u,v\} \subseteq V'} S_{uv}$, let $I_S = \{i \in [m]_+ \colon h_i \in S_l\}$, and let x_i and y_i denote the endvertices of k_i in K^i. Since $\mathscr{T}$ is n-close (hence, $\mathscr{T}_*$ is n-close), and since the weight of h_i is l, there is an x_iy_i-edge-cut D_{k_i} of size at most $n2^{n^3}$ in K^i, for each $i \in I_l$. Let $D_i = D_{k_i} - k_i$ for each $i \in I_l$. Now, let us essentially delete S_l from H' by deleting $D_l = \bigcup_{i \in I_l} D_i$ from $\mathscr{T}'$. Note that $|D_l| \leqslant \sum_{i \in I_l} |D_i| < |S_l| \cdot n2^{n^3} < n[c(c-1)/2]n2^{n^3} < 2^{n^3-1}c^2n^2$. So, after deleting fewer than $2^{n^3-1}c^2n^2$ edges from $\mathscr{T}'$, we may consider the collection $\mathfrak{T}' = \{\mathscr{T}'_{\backslash S_l}\} \cup \{\mathscr{T}_{\backslash D_{k_i}} \colon i \in I_S\}$ of edge-sum trees in place of $\mathscr{T}'$. Note that $G(\mathscr{T}') \cong_1 G(\mathfrak{T}')$. Let $\mathfrak{T}'' = (\mathfrak{T} - \{\mathscr{T}'\}) \cup \mathfrak{T}' = \{\mathscr{T}_{/F_s \backslash S_l}\} \cup \{\mathscr{T}_{/C_{k_i}} \colon i \in I_F\} \cup \{\mathscr{T}_{\backslash D_{k_i}} \colon i \in I_S\}$. It follows that $G(\mathscr{T})/C_s \backslash D_l \cong_1 G(\mathfrak{T}'')$. Note that each element of $\mathfrak{T}'' - \mathscr{T}_{/F_s \backslash S_l}$ is a $(d-1, N_n; n)$-edge-sum tree. Also, note that $\mathscr{T}'' = \mathscr{T}_{/F_s \backslash S_l}$ is a $(d,c;n)$-near-block tree. In fact, we show below that the root graph $H'' = H/F_s \backslash S_l$ of $\mathscr{T}''$ has only cycles of length less than c.

Let C' be a longest cycle in H different from C, if there is such a cycle. Since $\mathscr{T}$ is a block tree, H is 2-connected, and consequently, by Corollary 3.5, $|V(C) \cap V(C')| \geqslant 2$. Let $\{v_1, v_2\} \subseteq V(C) \cap V(C')$. If v_1 and v_2 are identified to the same vertex when F_s is contracted in H, then the subgraph of H' corresponding to C' is an edge-disjoint union of cycles of length of less than c. If v_1 and v_2 are identified to distinct vertices that we shall call v_1 and v_2, respectively, in H', then v_1 and v_2 belong to distinct components of H'' since $S_{v_1v_2}$ was essentially deleted from $\mathscr{T}'$ in forming $\mathscr{T}''$ (hence, $S_{v_1v_2}$ was deleted from H' in forming H''). It follows that $S_{v_1v_2} \cap E(C')$ is nonempty. So, the subgraph of H'' corresponding to C' contains fewer than c edges, and consequently C' is not a cycle of length c in H''. Hence, H'' has only cycles of length less than c.

We conclude that $\mathscr{T}''$ is a $(d, c-1; n)$-near-block tree if $c > 2$. Hence, $\mathfrak{T}''$ consists of one $(d, c-1; n)$-near-block tree and a number of $(d-1, N_n; n)$-edge-sum trees if $c > 2$. If $c = 2$, then each block of H'' contains at most one edge. Let I_1 and I_2 denote the subsets of $[m]_+$ so that $i \in I_1$ if and only if h_i is a loop in H'', and $i \in I_2$ if and only if h_i is a link-edge in H''. It follows that $G(\mathscr{T}'')$ is 1-isomorphic to $\bigcup_{i \in I_1} G(\mathscr{T}_i/k_i) \cup \bigcup_{i \in I_2} G(\mathscr{T}_i \backslash k_i)$. If $c > 2$, then let $\mathfrak{T}''_0 = \mathfrak{T}''$; if $c = 2$, then let $\mathfrak{T}''_0 = (\mathfrak{T}'' - \{\mathscr{T}''\}) \cup$

$\{\mathscr{T}_i/k_i\colon i \in I_1\} \cup \{\mathscr{T}_i \backslash k_i\colon i \in I_2\}$. It follows that $G(\mathscr{T})/C_s \backslash D_l \cong_1 G(\mathfrak{T}_0'')$. Note that $\mathfrak{T}_0''$ consists of a number of $(d-1, N_n; n)$-edge-sum trees and, if $c > 2$, one $(d, c-1; n)$-near-block tree.

If $t(G(\mathscr{T}/C_s \backslash D_l)) \leqslant 1$, then we have $t(G(\mathscr{T})) \leqslant |C_s| + |D_l| + t(G(\mathscr{T}/C_s \backslash D_l)) < c^3n^2/8 + 2^{n^3-1}c^2n^2 + 1$, in which case the lemma holds. So, we may assume that $t(G(\mathscr{T}/C_s \backslash D_l)) = t(G(\mathscr{T})/C_s \backslash D_l) > 1$. Then there is a 2-connected block B for which $t(B) = t(G(\mathscr{T}/C_s \backslash D_l))$. Since $G(\mathscr{T}/C_s \backslash D_l) \cong_1 G(\mathfrak{T}_0'')$, the block B is isomorphic to some block of $G(\mathscr{T}_0'')$, for some $\mathscr{T}_0'' \in \mathfrak{T}_0''$. By Lemma 7.3, there is a $(d, c-1; n)$-block tree or, if $c = 2$, a $(d-1, N_n; n)$-block tree whose composition is B. So, if we let $E_/ = C_s$ and $E_\backslash = D_l$, the proof is complete. $\square$

The next lemma is an extension of Lemma 7.6 in which some edges of the root graph of a $(d, c; n)$-close block tree may be unlabeled.

Lemma 7.7. *Let $\mathscr{T} = (\mathscr{G}, T)$ be a $(d, c; n)$-close block tree whose root graph contains no cycle of length exceeding c, for some integers n, c, and d exceeding* 3, 1, *and 0, respectively. Then one of the following holds*:

(i) *There are disjoint subsets $E_/$ and $E_\backslash$ of $E(G(\mathscr{T}))$ containing, respectively, fewer than $c^3n^2/8$ edges and fewer than $2^{n^3-1}c^2n^2$ edges, so that if B is a 2-connected block of $G(\mathscr{T})/E_/ \backslash E_\backslash$ whose type is $t(G(\mathscr{T})/E_/ \backslash E_\backslash)$, then there is a $(d, c-1; n)$-block tree if $c \geqslant 3$ or a $(d-1, N_n; n)$-block tree if $c = 2$, whose composition is B.*

(ii) $t(G(\mathscr{T})) < c^3n^2/8 + 2^{n^3-1}c^2n^2 + 1$.

(iii) $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$, *or* $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$.

Proof. We may assume that the root graph H of $\mathscr{T}$ contains at least one unlabeled edge; otherwise, the desired result is immediate, by Lemma 7.6. Let E_0 denote the set of unlabeled edges in H. For each $e \in E_0$, we can assign a direction and a new label ε_e to e, add a pendant link $\varepsilon_e = \xi\eta_e$ at the root ξ of the tree T of $\mathscr{T}$, let the node graph corresponding to η_e be a 2-cycle C_e, and assign a direction and the label ε_e to one of the edges of C_e. Let $\hat{\mathscr{T}}$ denote the resulting $(d, c; n)$-block tree, and let f_e denote the unlabeled edge of C_e, for each $e \in E_0$. It is evident that $G(\hat{\mathscr{T}}) \cong G(\mathscr{T})$. If $\hat{\mathscr{T}}$ is not n-close, then, by Corollary 6.3, an element of $\{C_{n,n-2}, C^*_{n,n-2}\}$ is a minor of $G(\hat{\mathscr{T}}) \cong G(\mathscr{T})$, and the result follows. So we may assume that $\hat{\mathscr{T}}$ is n-close.

If $t(G(\hat{\mathscr{T}})) < c^3n^2/8 + 2^{n^3-1}c^2n^2 + 1$, or if $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\hat{\mathscr{T}})$, then the lemma holds, since $G(\hat{\mathscr{T}}) \cong G(\mathscr{T})$. Otherwise, by Lemma 7.6, there are disjoint subsets $\hat{E}_/$ and $\hat{E}_\backslash$ of $E(G(\hat{\mathscr{T}}))$ containing, respectively, fewer than $c^3n^2/8$ edges and fewer than $2^{n^3-1}c^2n^2$ edges so that, if B is a 2-connected block of $G(\hat{\mathscr{T}})/\hat{E}_/ \backslash \hat{E}_\backslash$ for which $t(B) = t(G(\hat{\mathscr{T}})/\hat{E}_/ \backslash \hat{E}_\backslash)$, then there is a $(d, c-1; n)$-block tree if $c \geqslant 3$ or a $(d-1, N_n; n)$-block tree if $c = 2$, whose composition is B. Note that when Lemma 7.6 is applied to $\hat{\mathscr{T}}$, each edge in E_0 is weighted s in $\hat{\mathscr{T}}$, and the edges in $\hat{E}_\backslash$ correspond to edges of H weighted l in $\hat{\mathscr{T}}$. Let $E_/ = (\hat{E}_/ - \{f_e\colon e \in E_0\}) \cup \{e\colon f_e \in \hat{E}_/\}$ and $E_\backslash = \hat{E}_\backslash$.

It is straightforward that $|E_/| = |\hat{E}_/|$ and $|E_\backslash| = |\hat{E}_\backslash|$, that $E_/$ and $E_\backslash$ are disjoint subsets of $E(G(\mathscr{T}))$, and that $G((\mathscr{T})/E_/\backslash E_\backslash) \cong G(\hat{\mathscr{T}})/\hat{E}_/\backslash\hat{E}_\backslash$. The result follows. □

Now, we are ready to prove Theorem 7.1, whose proof uses several of the above lemmas.

Proof of Theorem 7.1. Let $\mathscr{T} = (\mathscr{G}, T)$ be a $(d, c; n)$-close 3-block tree. Recall that n is an integer exceeding 3, $0 \leqslant d \leqslant 2(n-1)$, and $2 \leqslant c \leqslant N_n$. We shall show that

$$t(G) \leqslant f(d) = d\sum_{i=1}^{N_n}\left(\frac{i^3n^2}{8} + 2^{n^3-1}i^2n^2\right) + \frac{N_n(N_n+1)}{2},$$

or $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$, or $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$. Note that

$$\begin{aligned} f(d) &= d\sum_{i=1}^{N_n}\left(\frac{i^3n^2}{8} + 2^{n^3-1}i^2n^2\right) + \frac{N_n(N_n+1)}{2} \\ &\leqslant 2(n-1)\left(\frac{n^2}{8}\sum_{i=1}^{N_n} i^3 + 2^{n^3-1}n^2\sum_{i=1}^{N_n} i^2\right) + \frac{N_n(N_n+1)}{2} \\ &< 2n\left(\frac{n^2N_n^2(N_n+1)^2}{8\cdot 4} + \frac{2^{n^3-1}n^2N_n(N_n+1)(2N_n+1)}{6}\right) + \frac{N_n(N_n+1)}{2} \\ &< \frac{n^3(N_n+1)^4}{16} + \frac{2^{n^3}n^3(N_n+1)^3}{3} + \frac{N_n(N_n+1)}{2} = F(n). \end{aligned}$$

We proceed by induction on d which includes within it induction on c. If $d = 0$, then, by Lemma 7.4, $t(G(\mathscr{T})) \leqslant c(c+1)/2 \leqslant N_n(N_n+1)/2 = 0\cdot\sum_{i=1}^{N_n}(i^3n^2/8 + 2^{n^3-1}i^2n^2) + N_n(N_n+1)/2$, as required. For the remainder of the proof, let us assume that $d > 0$, and the result holds for each $d' \in [d-1]$.

If the root graph of $\mathscr{T}$ is a cycle of length at least n, then, by Lemma 7.5, either $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$, or $t(G(\mathscr{T})) \leqslant n-2 < 2^n < N_n < f(d)$, or there are a set $S_\backslash$ of at most $n-3$ edges in $G(\mathscr{T})$ and a $(d-1, N_n; n)$-block tree $\mathscr{T}_B$ whose composition is a 2-connected block B of $G(\mathscr{T})\backslash S_\backslash$ for which $t(B) = t(G(\mathscr{T})\backslash S_\backslash)$. The first two of these alternatives imply the conclusion of the theorem, and so we may assume that the last condition listed holds. It follows that $B \leqslant_{\mathrm{m}} G(\mathscr{T})$ and that $t(G(\mathscr{T})) \leqslant |S_\backslash| + t(B)$. If $\mathscr{T}_B$ is not n-close, then, by Corollary 6.3, $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$ or $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$, and the conclusion follows. So, we may assume that $\mathscr{T}_B$ is n-close. By the induction hypothesis, $t(B) \leqslant f(d-1)$. It follows that $t(G(\mathscr{T})) \leqslant n-3+f(d-1) < \sum_{i=1}^{N_n}(i^3n^2/8 + 2^{n^3-1}i^2n^2) + f(d-1) = f(d)$, as required.

We may assume for the remainder of the proof that the root graph of $\mathscr{T}$ contains no cycles of length exceeding c. By Lemma 7.7, either a graph in $\{C_{n,n-2}, C^*_{n,n-2}\}$ is a minor of $G(\mathscr{T})$, or $t(G(\mathscr{T})) < c^3n^2/8 + 2^{n^3-1}c^2n^2 + 1 < N_n^3n^2/8 + 2^{n^3-1}N_n^2n^2 + N_n(N_n+1)/2 < f(d)$, or conclusion (i) in Lemma 7.7 holds. The first two of these alternatives imply the conclusion of the theorem, and so we may assume that the last condition listed holds.

Now, we shall show that $t(G(\mathscr{T})) \leqslant g(c) = \sum_{i=1}^{c}(i^3n^2/8 + 2^{n^3-1}i^2n^2) + f(d-1)$, or $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$, or $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$. Note that $g(c) = \sum_{i=1}^{c}(i^3n^2/8 + 2^{n^3-1}i^2n^2) + (d-1)\sum_{i=1}^{N_n}(i^3n^2/8 + 2^{n^3-1}i^2n^2) + N_n(N_n+1)/2 \leqslant f(d)$. Hence, it will follow that $t(G(\mathscr{T})) \leqslant f(d)$, or $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$, or $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$. If $c=2$, then there are disjoint sets $E_/$ and $E_\backslash$ containing fewer than n^2 and $2^{n^3+1}n^2$ edges, respectively, in $G(\mathscr{T})$ and a $(d-1, N_n; n)$-block tree $\mathscr{T}_B$ whose composition is a 2-connected block B of $G(\mathscr{T})/E_/\backslash E_\backslash$ such that $t(B) = t(G(\mathscr{T})/E_/\backslash E_\backslash)$. If $\mathscr{T}_B$ is not n-close, then, by Corollary 6.3, $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$ or $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$, and the conclusion follows. So, we may assume that $\mathscr{T}_B$ is n-close. By the induction hypothesis, $t(B) \leqslant f(d-1)$. It follows that $t(G(\mathscr{T})) \leqslant |E_/| + |E_\backslash| + t(B) < n^2 + 2^{n^3+1}n^2 + f(d-1) < \sum_{i=1}^{2}(i^3n^2/8 + 2^{n^3-1}i^2n^2) + f(d-1) = g(2) \leqslant f(d)$, as required. So, let us assume that $3 \leqslant c \leqslant N_n$ and that $t(G(\mathscr{U})) \leqslant g(c')$, or $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{U})$, or $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{U})$ when $\mathscr{U}$ is a $(d, c'; n)$-close block tree and c' satisfies $2 \leqslant c' < c$.

There are disjoint subsets $E_/$ and $E_\backslash$ of $G(\mathscr{T})$ containing fewer than $c^3n^2/8$ edges and $2^{n^3-1}c^2n^2$ edges, respectively, and a $(d, c-1; n)$-block tree T_B whose composition is a 2-connected block B of $G(\mathscr{T})/E_/\backslash E_\backslash$ such that $t(B) = t(G(\mathscr{T})/E_/\backslash E_\backslash)$, by Lemma 7.7. Again, by Corollary 6.3, if $\mathscr{T}_B$ is not n-close as before, $C_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$ or $C^*_{n,n-2} \leqslant_{\mathrm{m}} G(\mathscr{T})$, and the claim holds. So, we may assume that $\mathscr{T}_B$ is n-close. By the second induction hypothesis, $t(B) \leqslant g(c-1)$. It follows that $t(G(\mathscr{T})) \leqslant |E_/| + |E_\backslash| + t(B) < c^3n^2/8 + 2^{n^3-1}c^2n^2 + g(c-1) = g(c) \leqslant f(d)$, as required. The theorem follows. $\square$

Since each 2-connected graph with more than two edges can be decomposed into a unique 3-block tree, and since a 2-connected graph with at most 2 edges has very small type, the theorem below follows immediately on combining Corollary 4.4, Theorem 5.1, and Theorem 7.1.

Theorem 7.8. *If G is a 2-connected graph such that*

$$t(G) \geqslant \frac{n^3(N_n+1)^4}{16} + \frac{2^{n^3}n^3(N_n+1)^3}{3} + \frac{N_n(N_n+1)}{2}$$

for some integer n exceeding 3, then an element of $\{F_n, C_{n,n-2}, C^*_{n,n-2}\}$ is a minor of G. $\square$

Since the type of a general graph is the maximum of types of its blocks, Theorem 7.8 immediately implies Theorem 1.7.

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